Uncovering the Fibonacci Phase

in Z_3 Parafermion Systems

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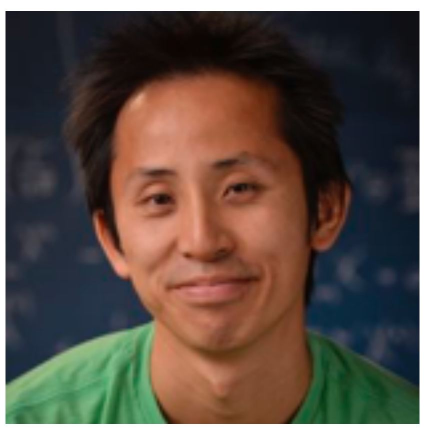
University of Virginia 2015

Collaborators:

David Clarke - Caltech / Maryland

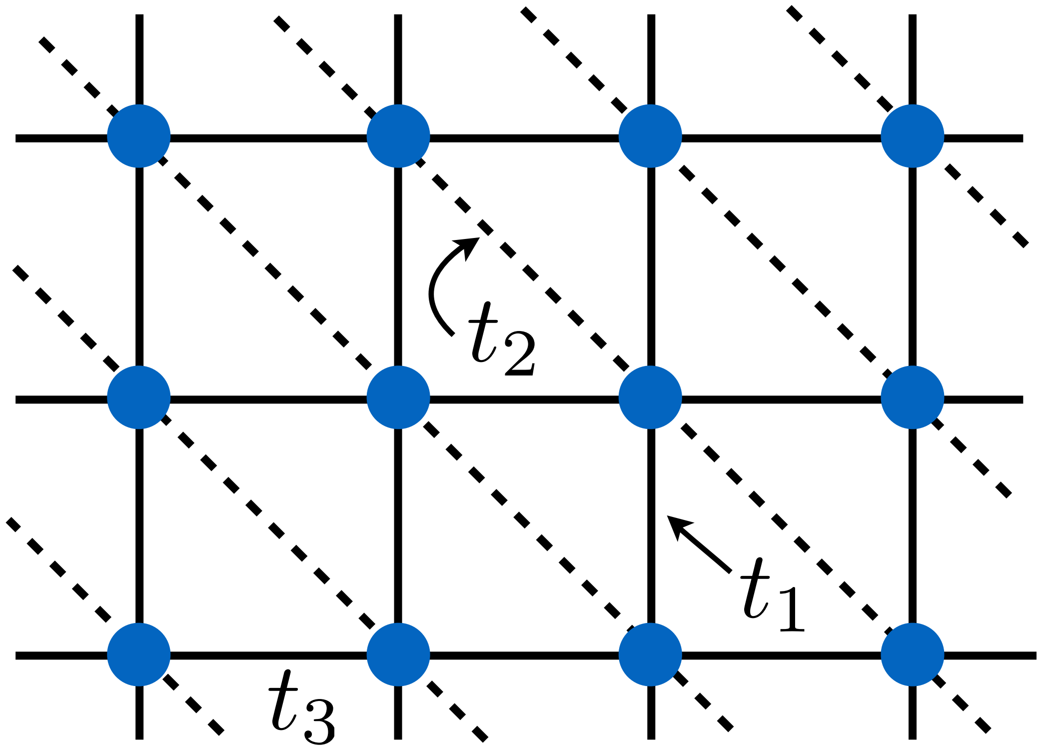


Roger Mong - Caltech / Pittsburgh

Jason Alicea - Caltech

In this talk:

- Two-dimensional lattice model containing square lattice, triangular lattice, and decoupled chain limits



- Site degrees of freedom are "parafermions"

- Strong evidence for emergent **Fibonacci anyon** quasiparticle on **isotropic triangular lattice** (and likely "t_1-t_2" model as well)

Technique used in this talk is
the *density matrix renormalization group* (DMRG)

Works by "compressing" many-body wavefunction

$$|\Psi\rangle = [\bullet]$$

d^N components

DMRG:

$\sim Nd$ components

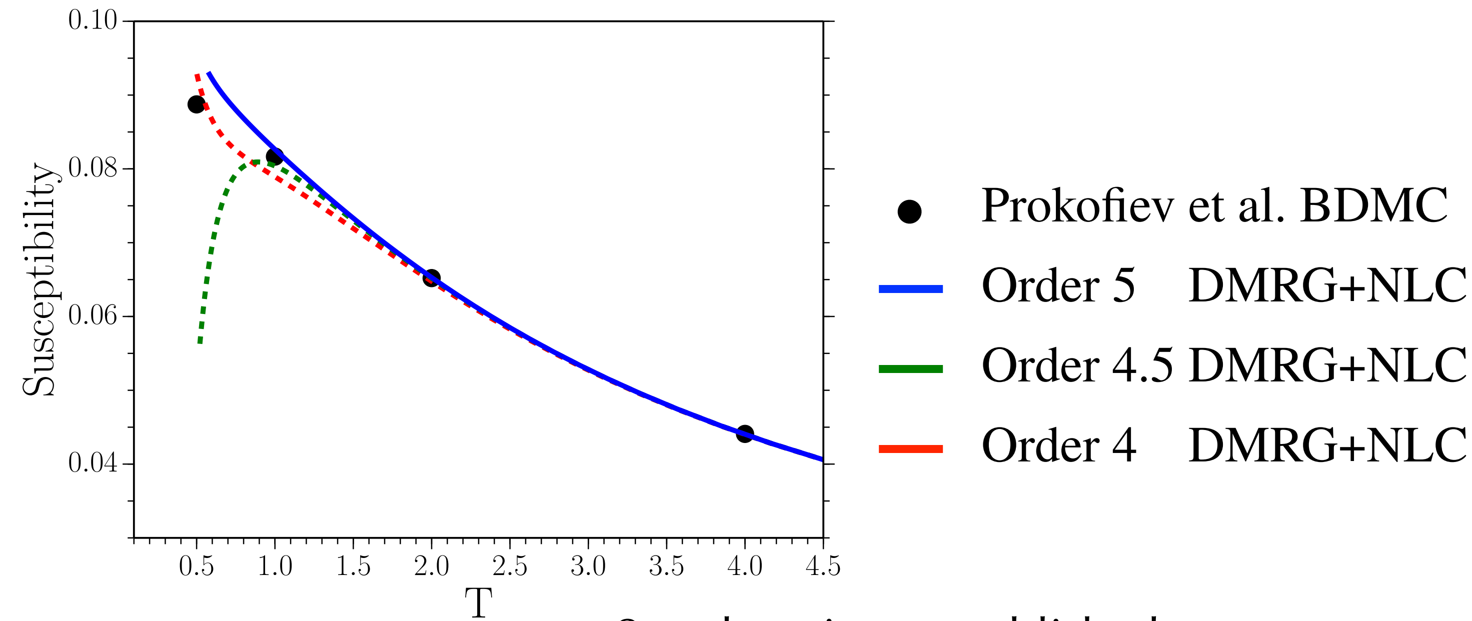
$$|\tilde{\Psi}\rangle = \begin{bmatrix}\bullet & \bullet \\ \bullet & \bullet\end{bmatrix}\begin{bmatrix}\bullet & \bullet \\ \bullet & \bullet\end{bmatrix}\begin{bmatrix}\bullet & \bullet \\ \bullet & \bullet\end{bmatrix}\begin{bmatrix}\bullet & \bullet \\ \bullet & \bullet\end{bmatrix}\cdots\begin{bmatrix}\bullet & \bullet \\ \bullet & \bullet\end{bmatrix}$$

Mean field:

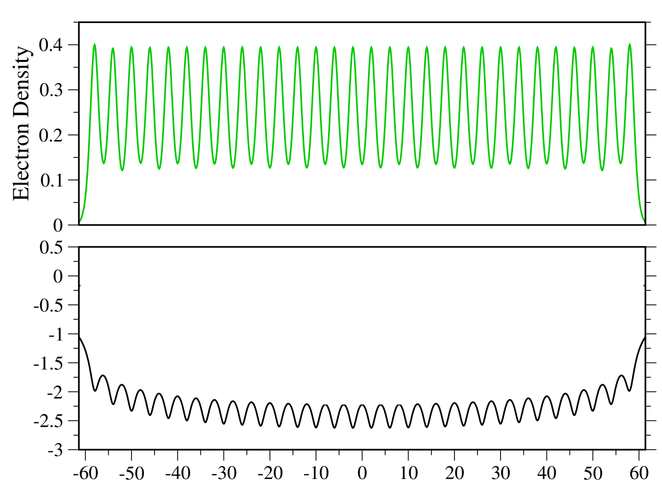
$$\Big(|\tilde{\Psi}_{\mathrm{MF}}\rangle = [\bullet]\,[\bullet]\,[\bullet]\,[\bullet]\cdots[\bullet]\Big)$$

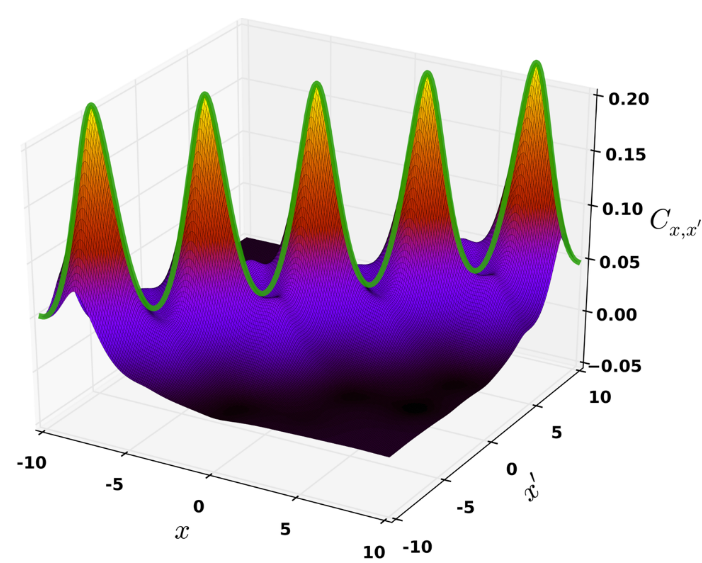
DMRG can address wide variety of systems

Frustrated magnets
Infinite, finite T
triangular lattice
Heisenberg model:



Stoudenmire, unpublished





Fermions
e.g. continuum 1d
systems

Lattice models of ‘**anyons**’ in two dimensions...

A major goal of 21st century physics:
build a scalable quantum computer

Ingredients:

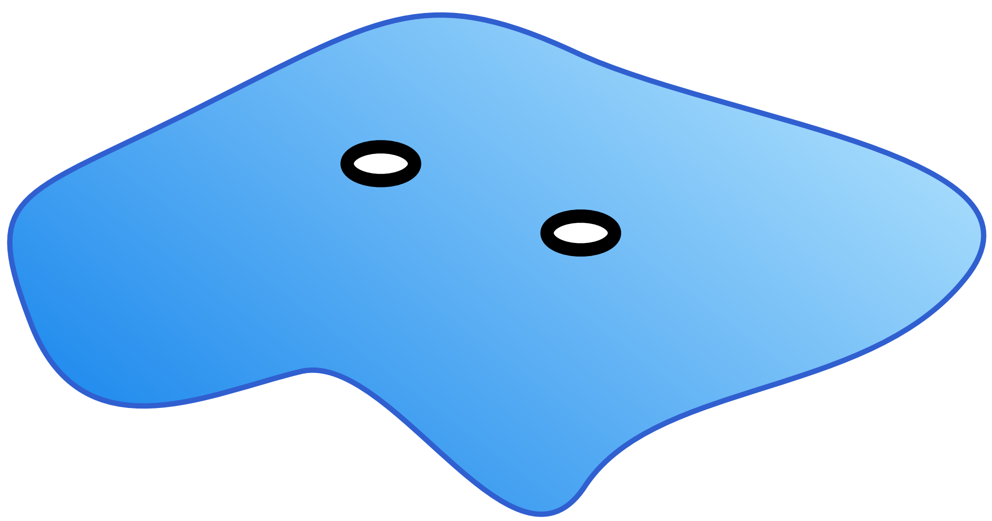
- -Qubits (e.g. a spin 1/2)
- -Unitary operations on these qubits

Challenges:

- -Stability (decoherence)
- -Usefulness (universal computation?)

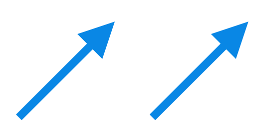
Promising approach for dealing with decoherence is *topological quantum computing*

In certain topological phases qubit space can be 'hidden'



$\sim$ 

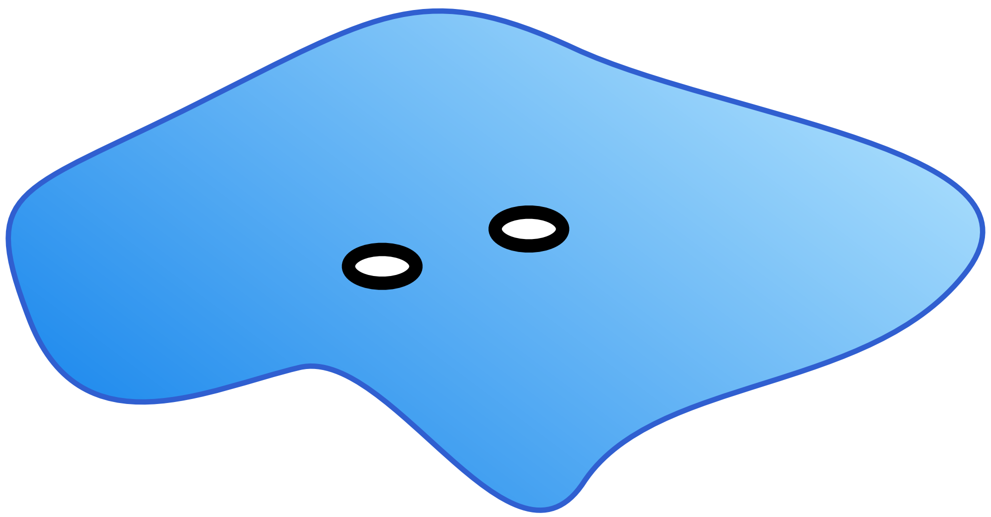
two-level system*

(cf. two spin 1/2's  $\sim$ four-level system)

Information stored non-locally: **decoherence protection**

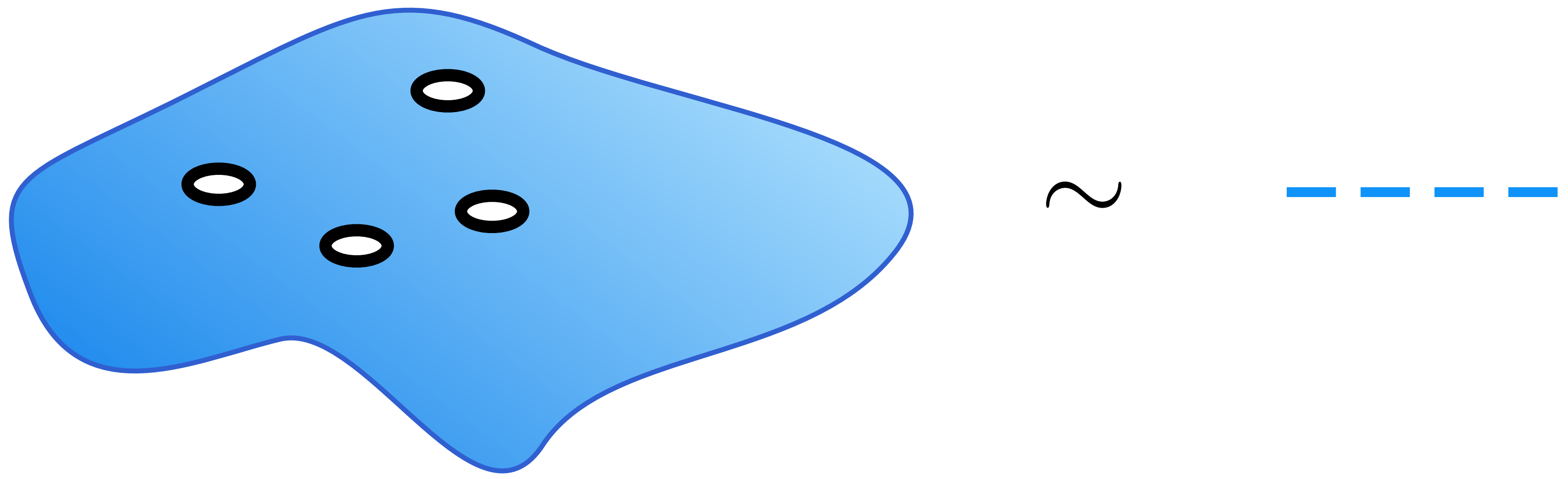
*for Majorana fermion case

More quasiparticles → additional qubits

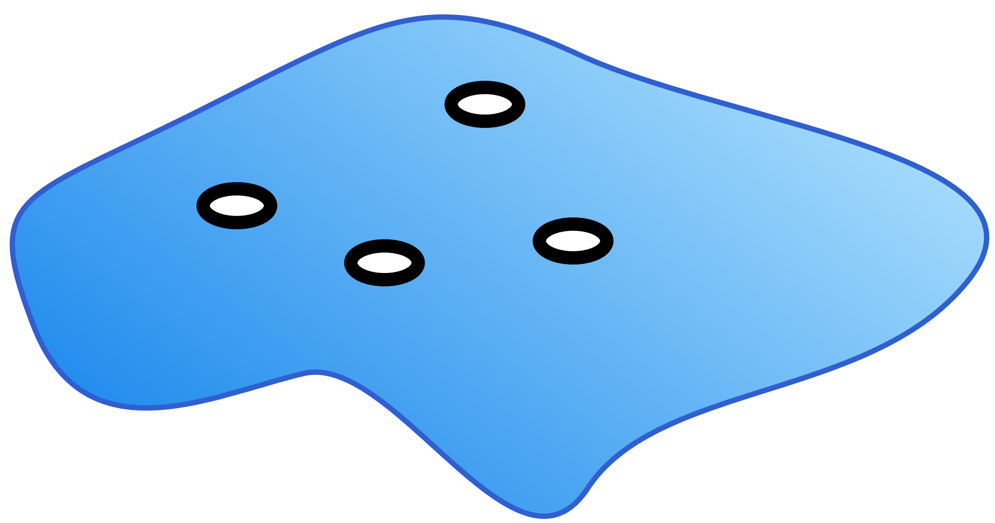


$\sim$

More quasiparticles → additional qubits



More quasiparticles $\longrightarrow$ additional qubits

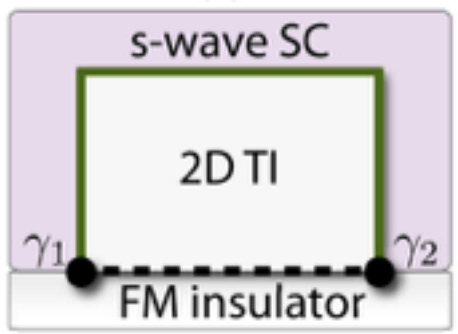


$$\sim \hat{U}$$

Qubits can be manipulated by ‘braiding’ quasiparticles

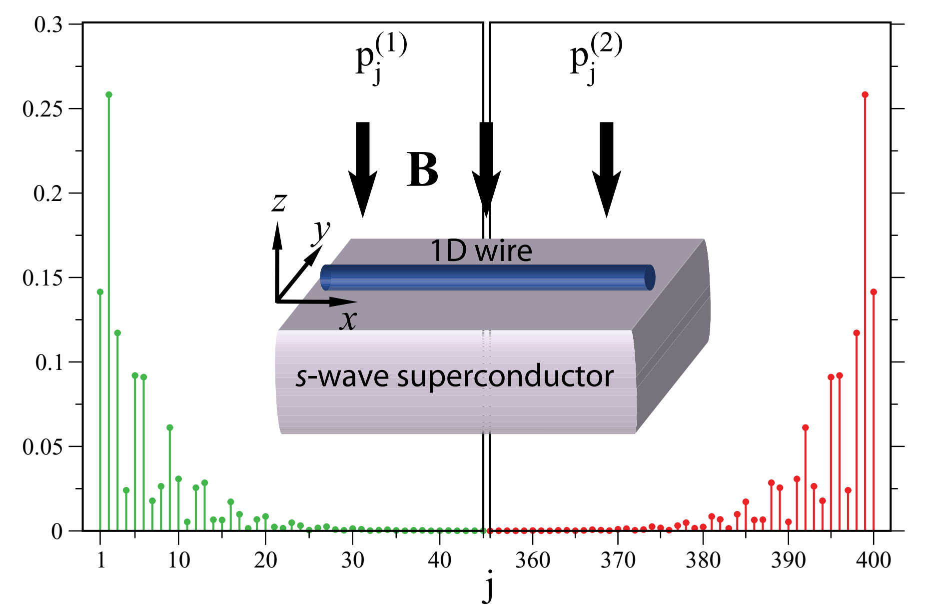
Encouraging progress in *engineering* such "non-Abelian anyons"

Microscopic platforms for Majorana zero modes



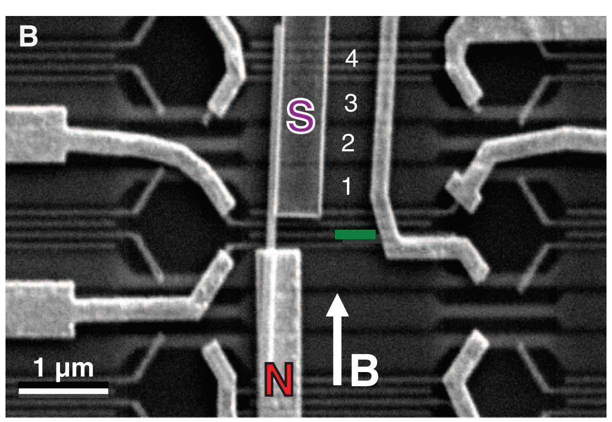
Fu, Kane PRB 79, 161408(R) (2009)

Lutchyn et al. PRL 105, 077001 (2010)
Oreg et al. arxiv:1003.1145 (2010)

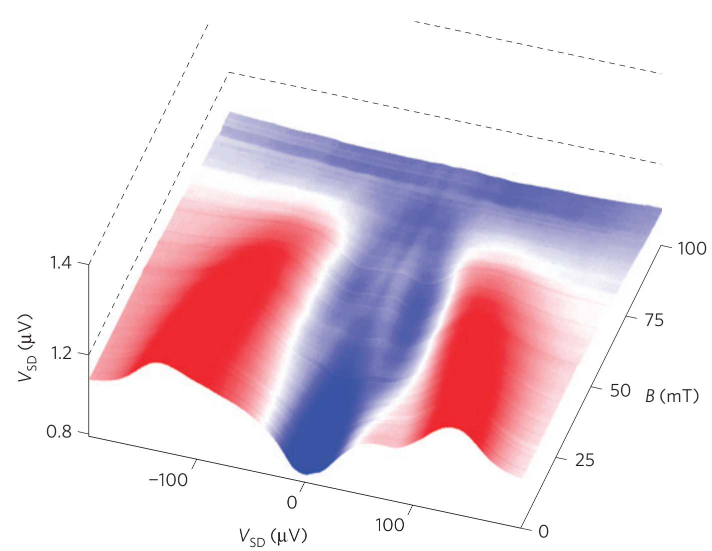


Stoudenmire, Alicea, Starykh, Fisher PRB 84, 014503 (2011)

Experimental realization?



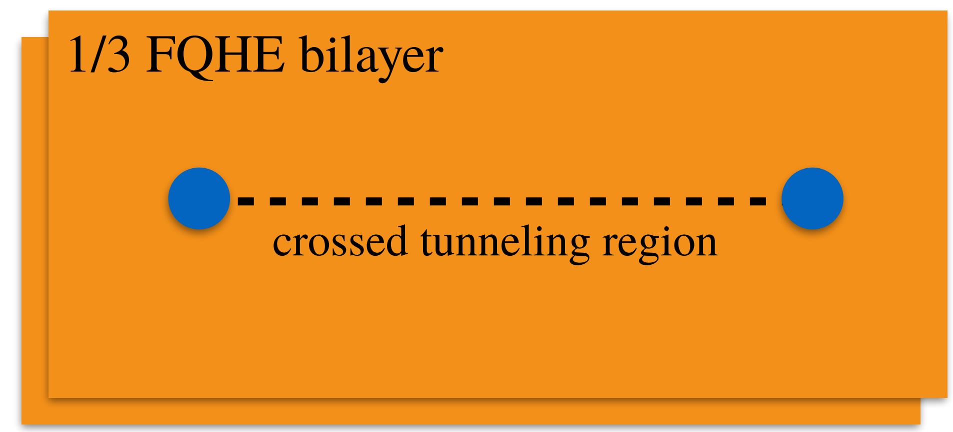
V. Mourik et al., Science 336, 1003 (2012).



A. Das et al., Nat. Phys. 8, 887 (2012).

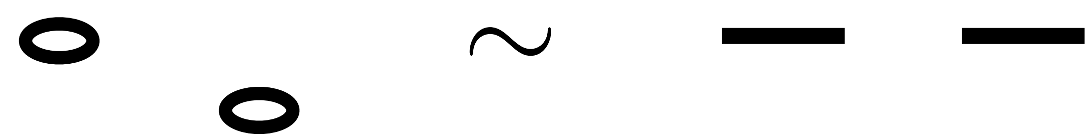
New platforms under way for *parafermions*, simplest generalization of Majorana fermions

Clarke, Alicea, and Shtengel, Nat. Commun. 4, 1348 (2013)



Barkeshli and Qi, PRX 2, 031013 (2012)

Majorana:



2-level system

Z3 Parafermion:

3-level system

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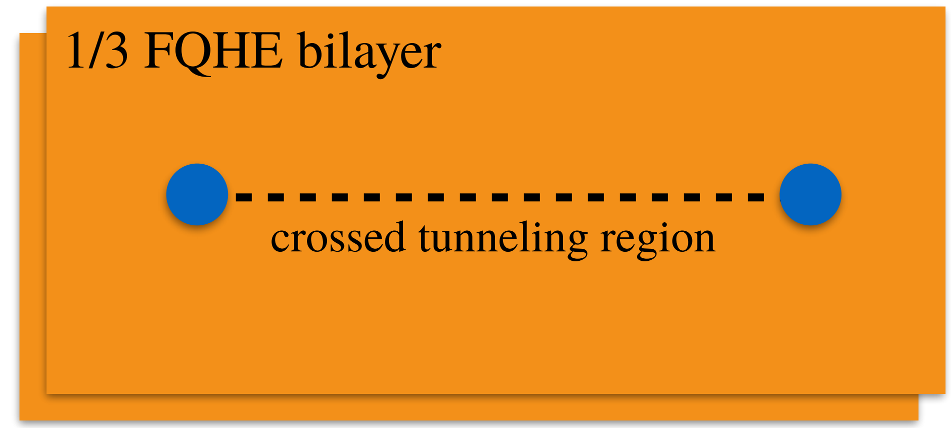
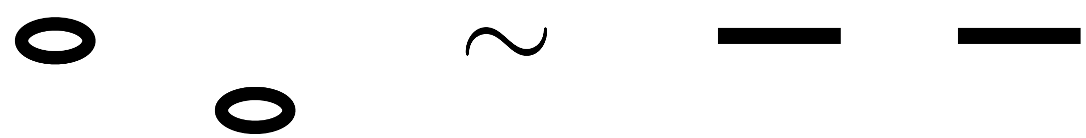
Clarke, Alicea, and Shtengel, Nat. Commun. 4, 1348 (2013)



Barkeshli and Qi, PRX 2, 031013 (2012)

Majorana:



2-level system

Z3 Parafermion:*

3-level system

*also different commutation relations from Majorana

New platforms under way for *parafermions*, simplest generalization of Majorana fermions

Clarke, Alicea, and Shtengel, Nat. Commun. 4, 1348 (2013)

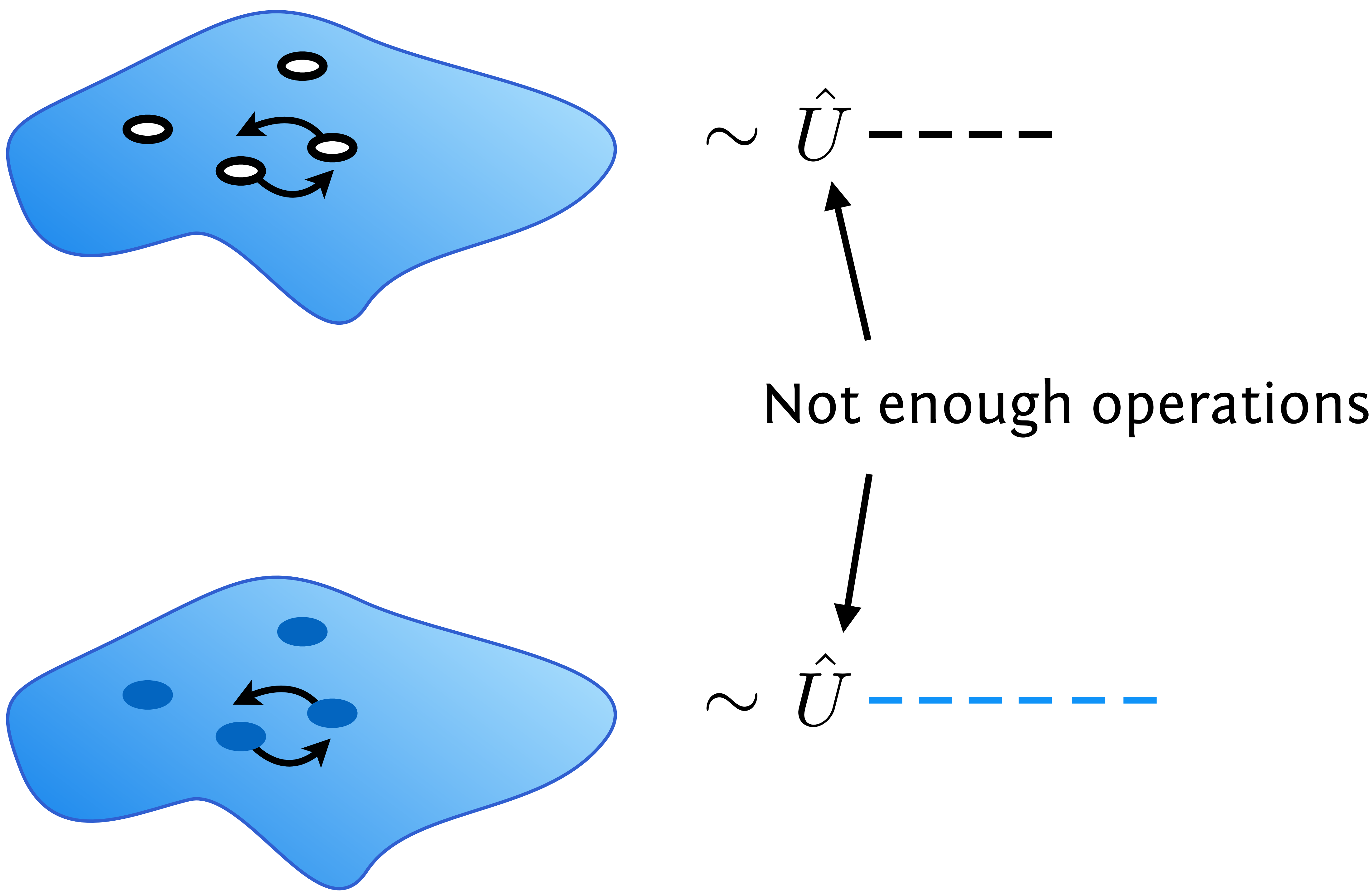
Barkeshli and Qi, PRX 2, 031013 (2012)

[Needs development]

Schemes will continue to improve....

But why pursue them?

Because Majorana fermions & parafermions
insufficient for <u>universal</u> <u>quantum</u> <u>computation</u>

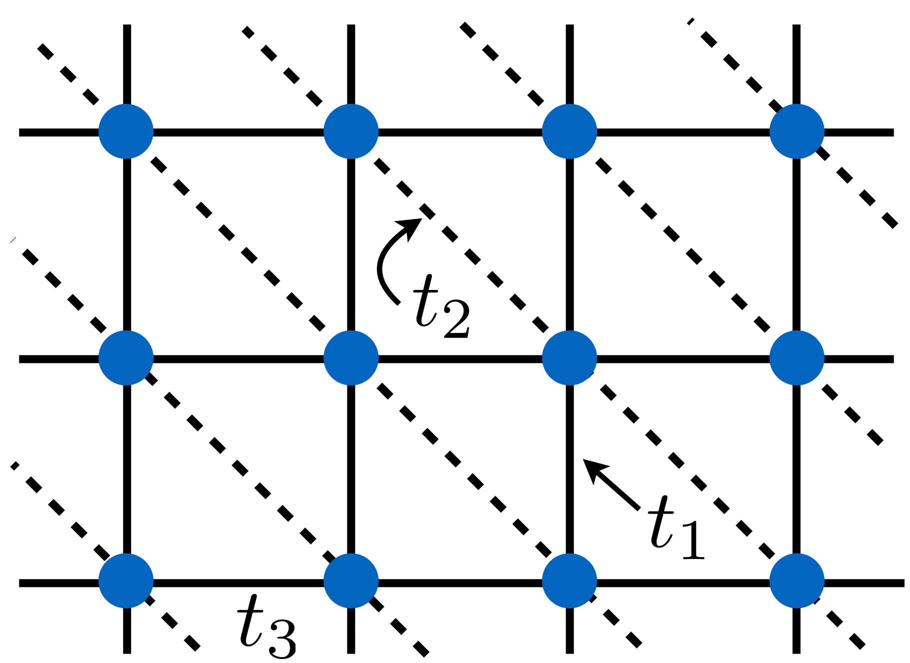


Yet parafermions may hold the key...

This talk:

parafermions could hybridize to yield

Fibonacci anyons



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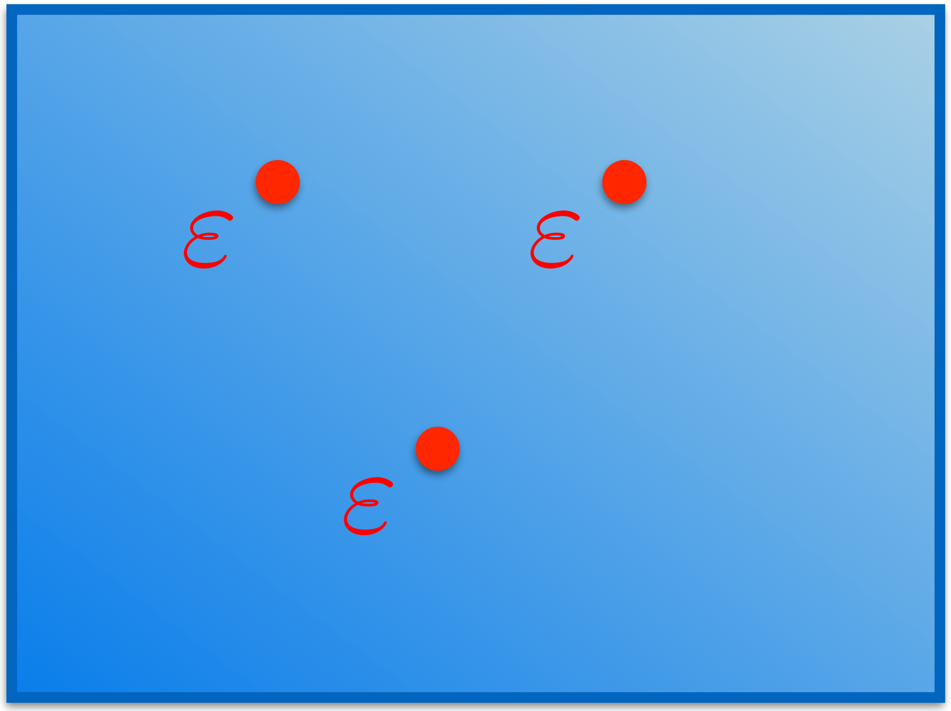


Unlike parafermion

 $\sim$ 

3-level system

Fibonacci anyons

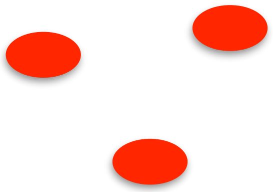
 $\sim$ 1 level system

Unlike parafermion

 $\sim$ 

3-level system

Fibonacci anyons

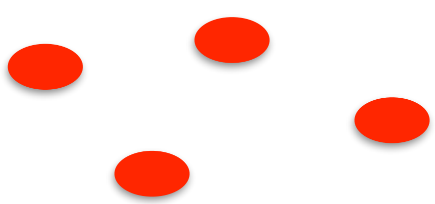
 $\sim$ 1, 1 level system

Unlike parafermion

 $\sim$ 

3-level system

Fibonacci anyons

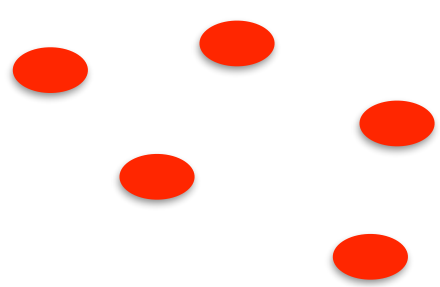
 $\sim$ 1, 1, 2 level system

Unlike parafermion

 $\sim$ 

3-level system

Fibonacci anyons

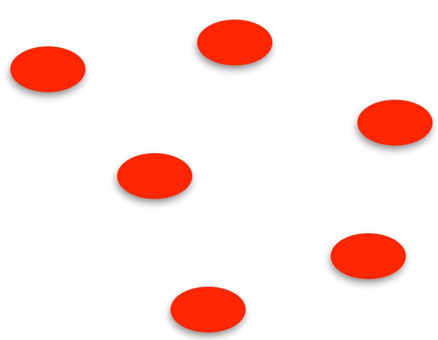
 $\sim$ 1, 1, 2, 3 level system

Unlike parafermion

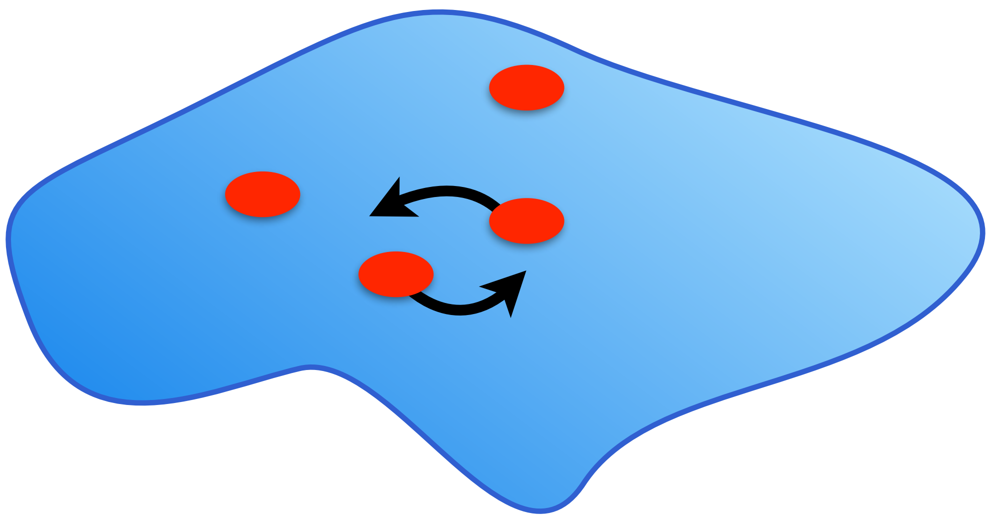
 $\sim$ 

3-level system

Fibonacci anyons

 $\sim$ 1, 1, 2, 3, 5 level system

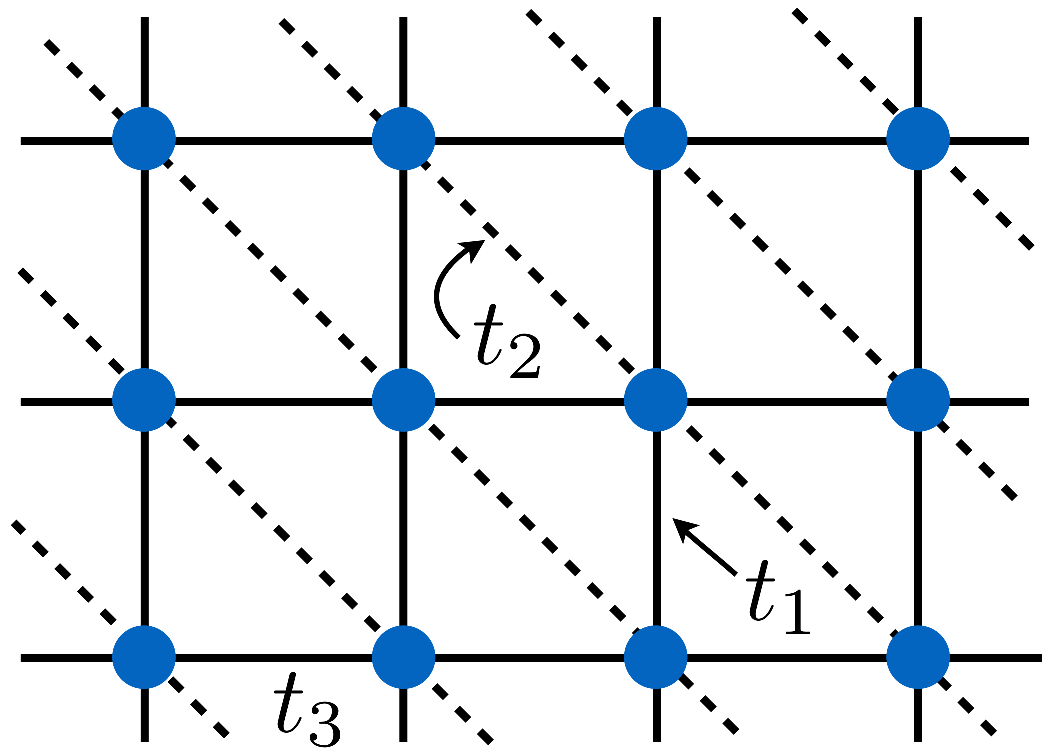
More importantly, Fibonacci quasiparticles have universal braiding



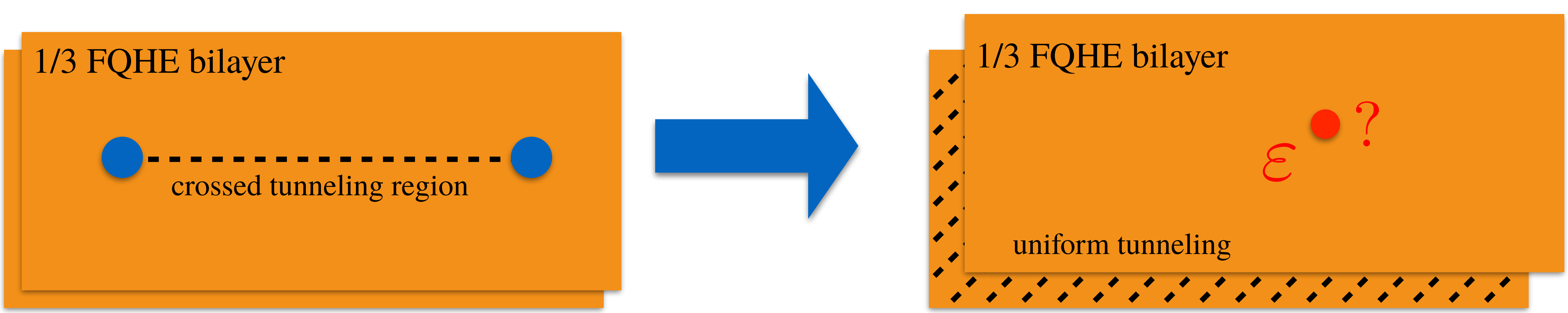
$$\sim \hat{U} \quad — \quad —$$

enough operations for quantum computing

Finally, parafermion lattice model just a "crutch"

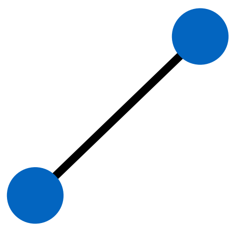


"Smeared" limit could be sufficient for Fibonacci*



*Barkeshli, Vaezi PRL 113, 236804 (2014). Also see: Liu et al. arxiv:1502.05391; Geraedts et al. arxiv:1502.01340 for negative result

Hybridizing Parafermions



Warmup #1: parafermion dimer

$$H = -\frac{t}{2}(\omega \alpha_i^\dagger \alpha_j + \bar{\omega} \alpha_j^\dagger \alpha_i) \qquad [\ \omega = e^{i2\pi/3}\ ]$$

Simplest parafermion Hamiltonian

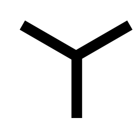
Strongly interacting, despite appearance

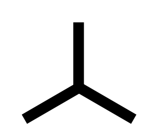
Warmup #1: parafermion dimer

$$H = -\frac{t}{2}(\omega \alpha_i^\dagger \alpha_j + \bar{\omega} \alpha_j^\dagger \alpha_i) \qquad [\ \omega = e^{i2\pi/3}\ ]$$

Hamiltonian (by mapping to ‘clock’ variables):

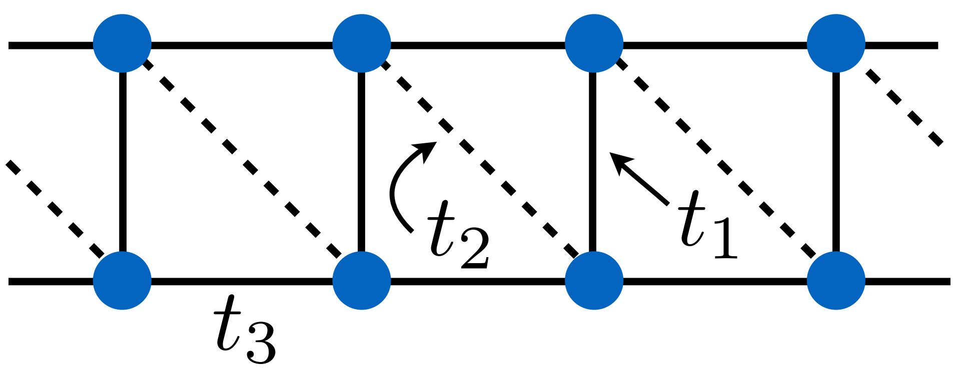
$$\begin{bmatrix} t/2 & & \\ & -t & \\ & & t/2 \end{bmatrix}$$

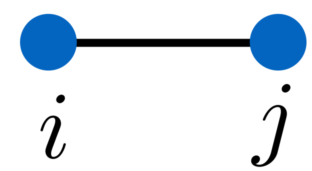
Positive $t > 0$, unique ground state 

Negative $t < 0$, two ground states 

Sign of t important!

Warmup #2: two-leg ladder





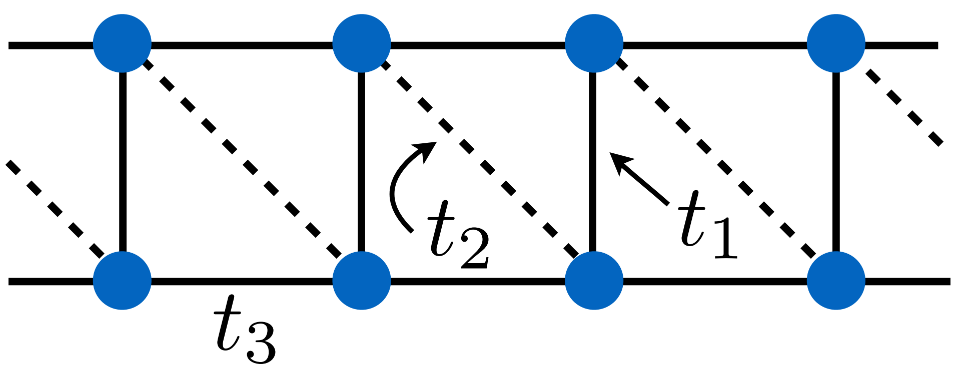
$$= -t\left(\omega \alpha_i^\dagger \alpha_j + \mathrm{H.c}\right)$$

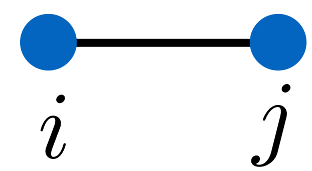
'Squeezed' system will 'point' us toward 2d Fibonacci phase

Can understand in two limits:

- $t_1 \gg t_2, t_3$
- $t_2 \gg t_1, t_3$

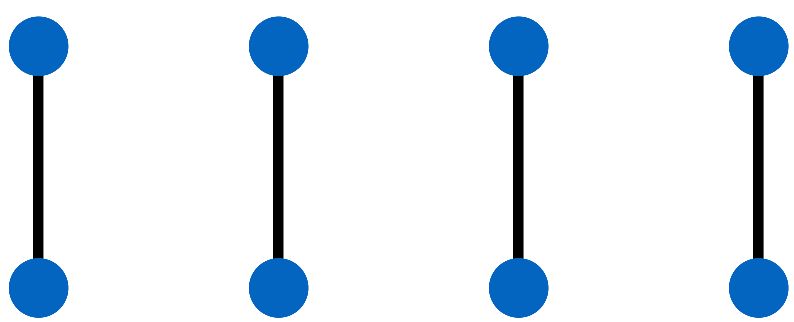
Warmup #2: two-leg ladder





$= -t\left(\omega\alpha_i^\dagger\alpha_j + \mathrm{H.c}\right)$

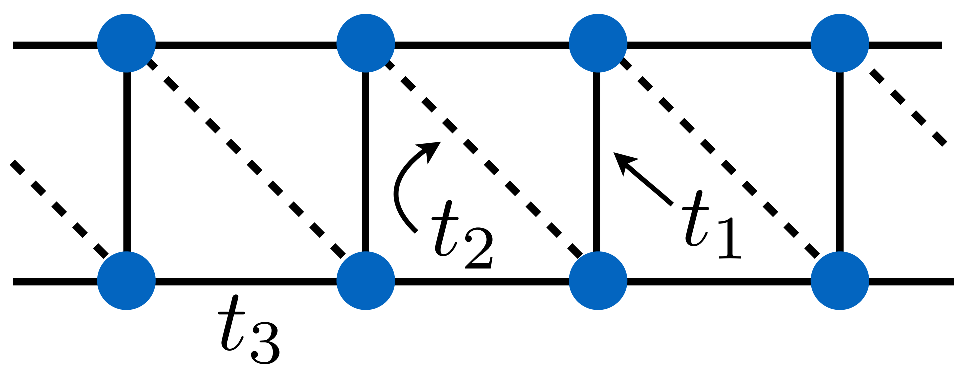
$$t_1 \gg t_2, t_3$$

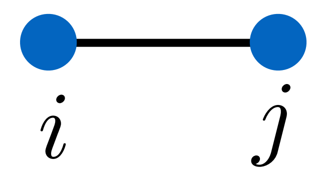


Parafermions "pair" along rungs

Remain in trivial gapped phase for small t_2, t_3

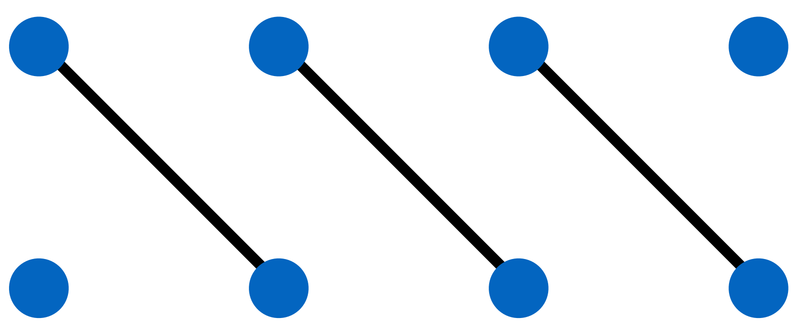
Warmup #2: two-leg ladder





$$= -t\left(\omega \alpha_i^\dagger \alpha_j + \mathrm{H.c}\right)$$

$$t_2 \gg t_1, t_3$$

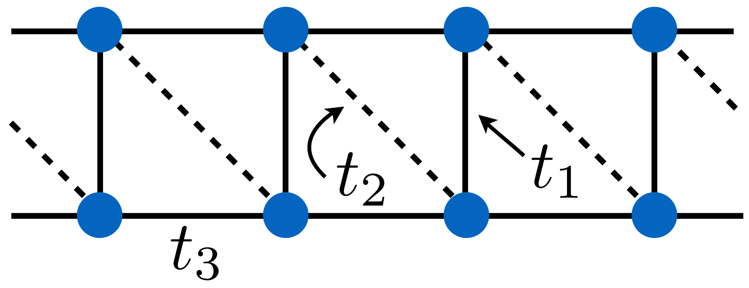


Parafermions “pair” diagonally

Fractionalized 3-fold degenerate edge state

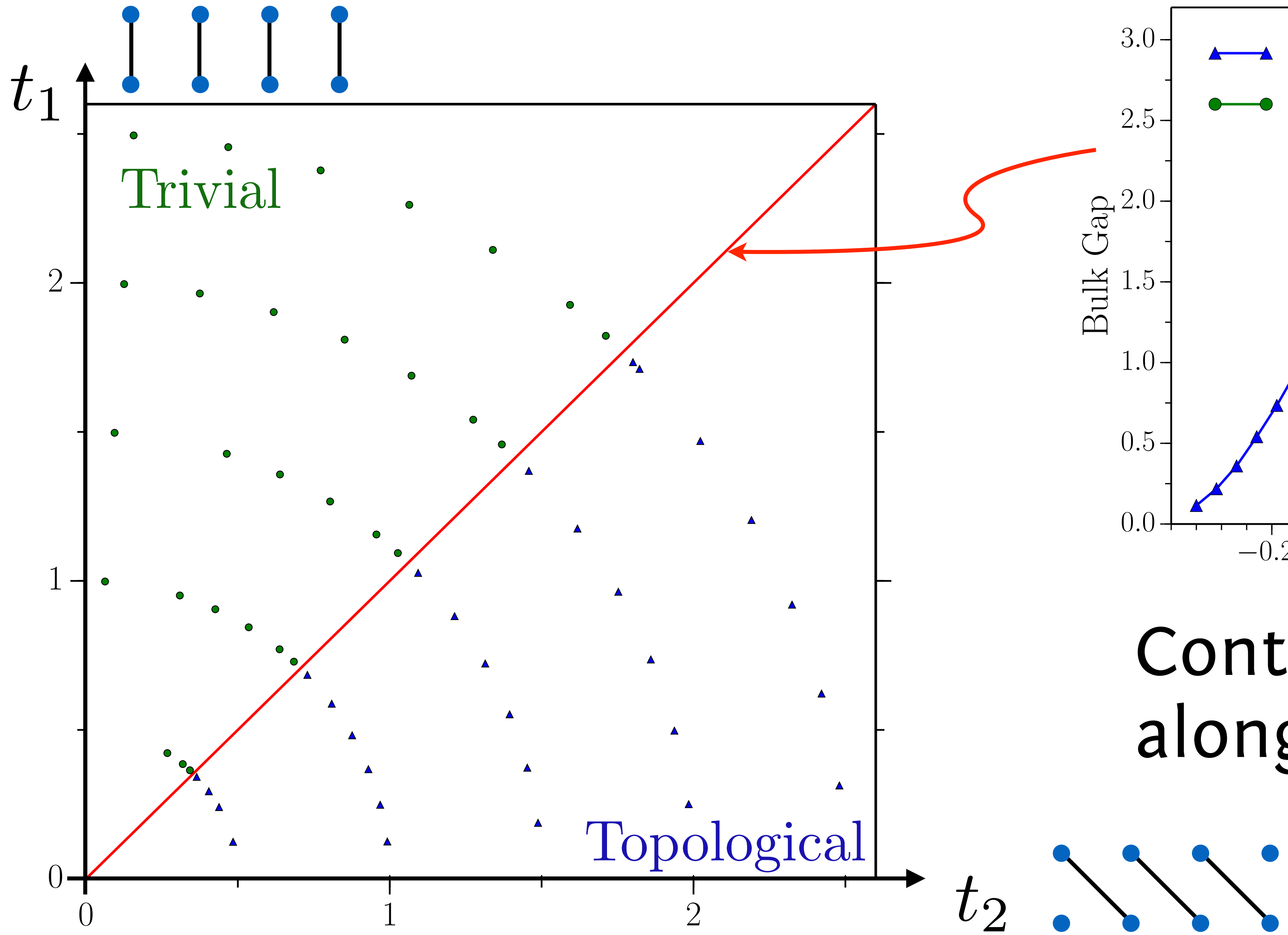
Remain in topological phase for small t_1, t_3

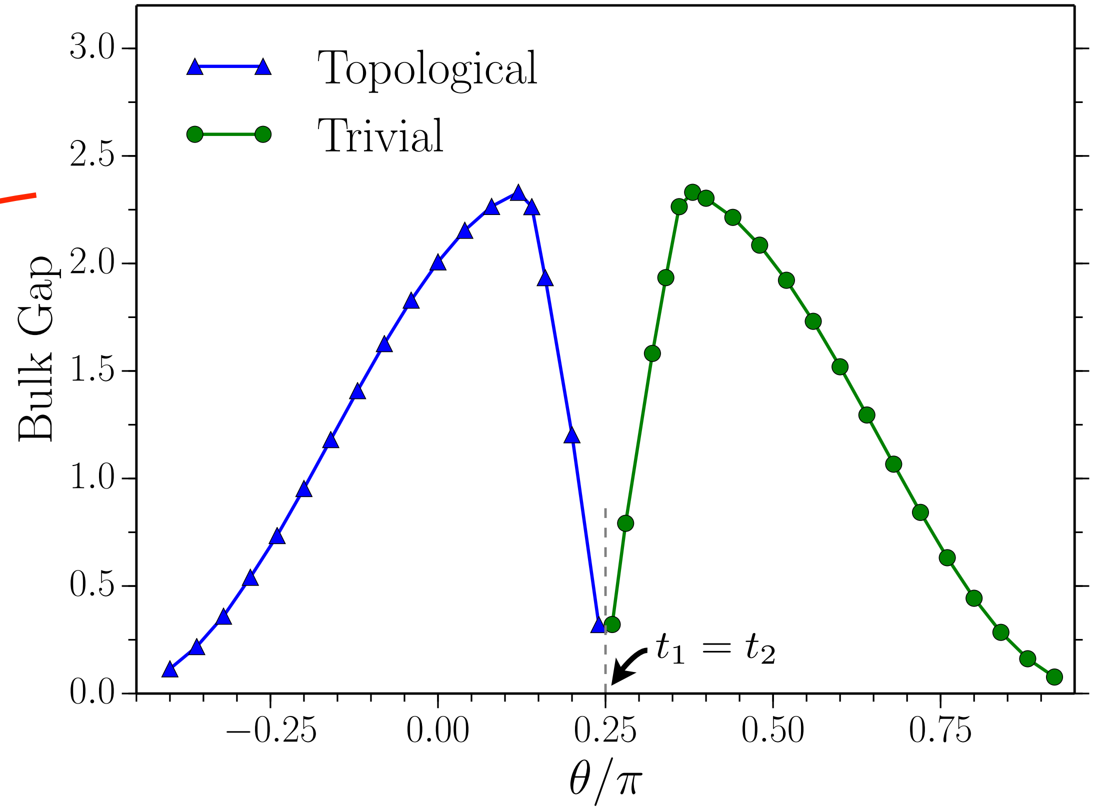
Warmup #2: two-leg ladder



Phases compete for $t_1 \approx t_2$

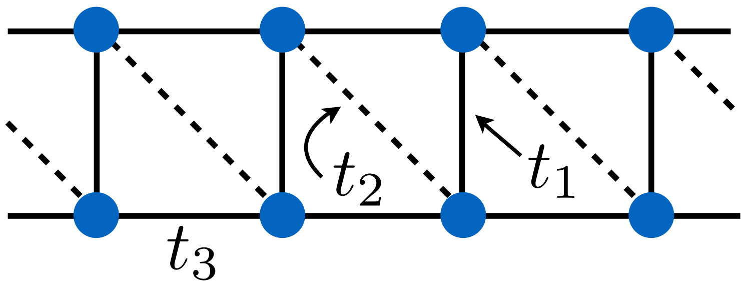
DMRG results for phase boundary:





Continuous transition along $t_1 = t_2$ line

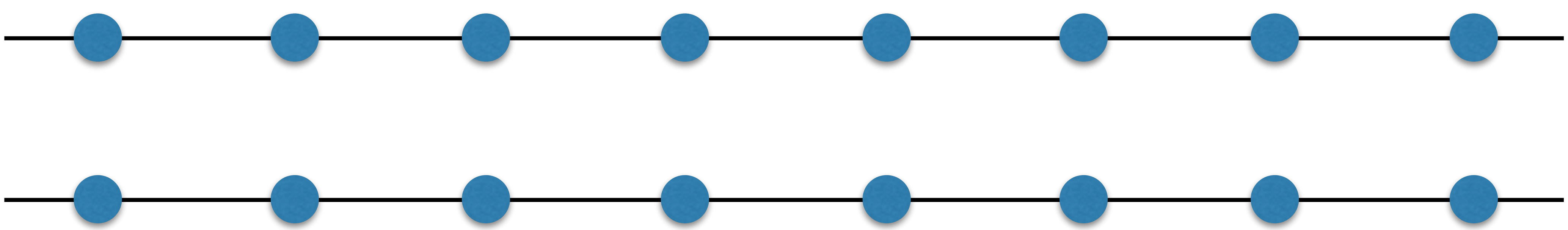
Warmup #2: two-leg ladder



Duality argument shows transition exactly at $t_1 = t_2$!

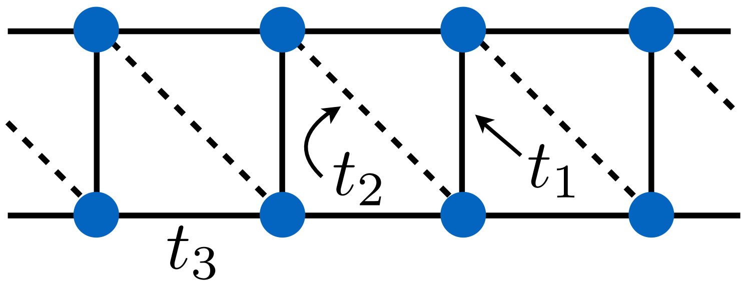
Suggestive field theory picture

For $t_1 = t_2 = 0$, $t_3 > 0$, each chain described by ‘Z_3 parafermion’ conformal field theory (CFT)[1,2]



1) Zamolodchikov, Fateev Phys. Lett. A 92, 37 (1982).
2) Fendley J. Stat. Mech. (2012) P11020.

Warmup #2: two-leg ladder



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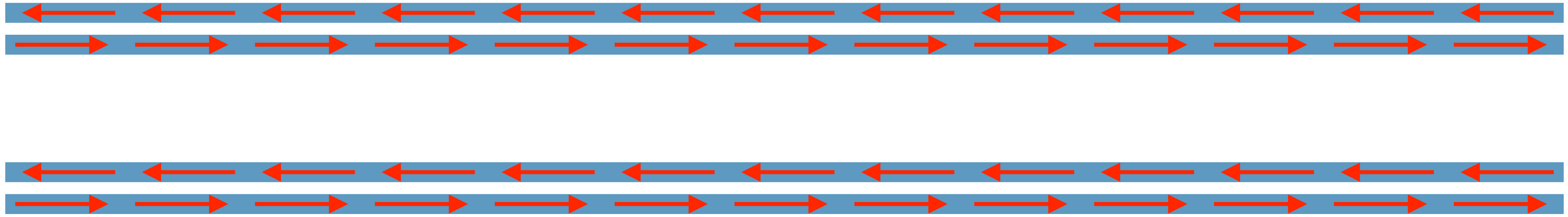
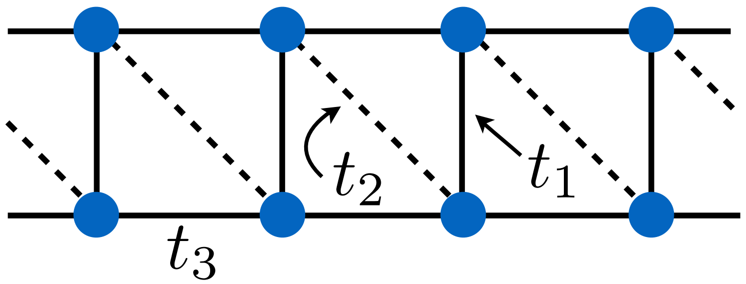
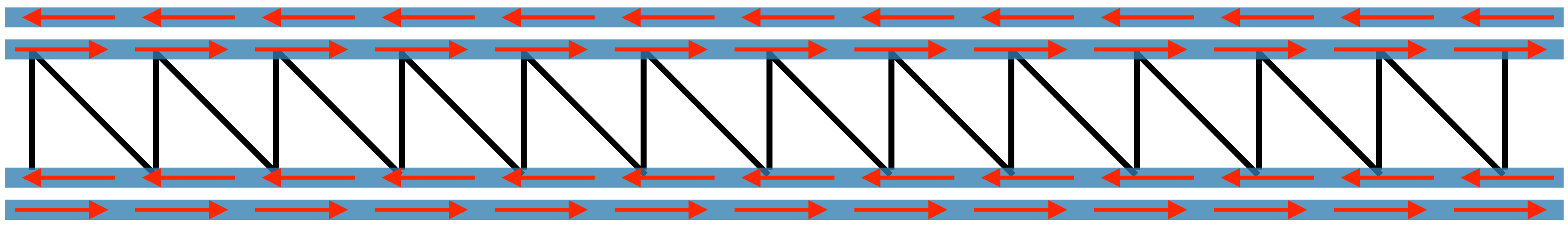


1) Zamolodchikov, Fateev Phys. Lett. A 92, 37 (1982).
2) Fendley J. Stat. Mech. (2012) P11020.

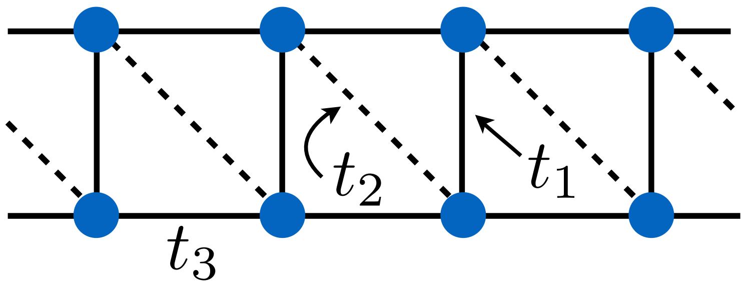
Warmup #2: two-leg ladder



Fine tuning $0 < t_1 = t_2 \ll 1$ couples only left mover of bottom chain to right mover of top chain

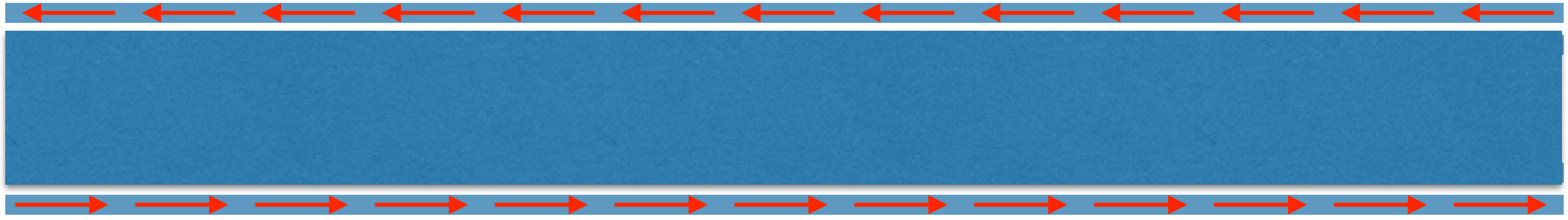


Warmup #2: two-leg ladder

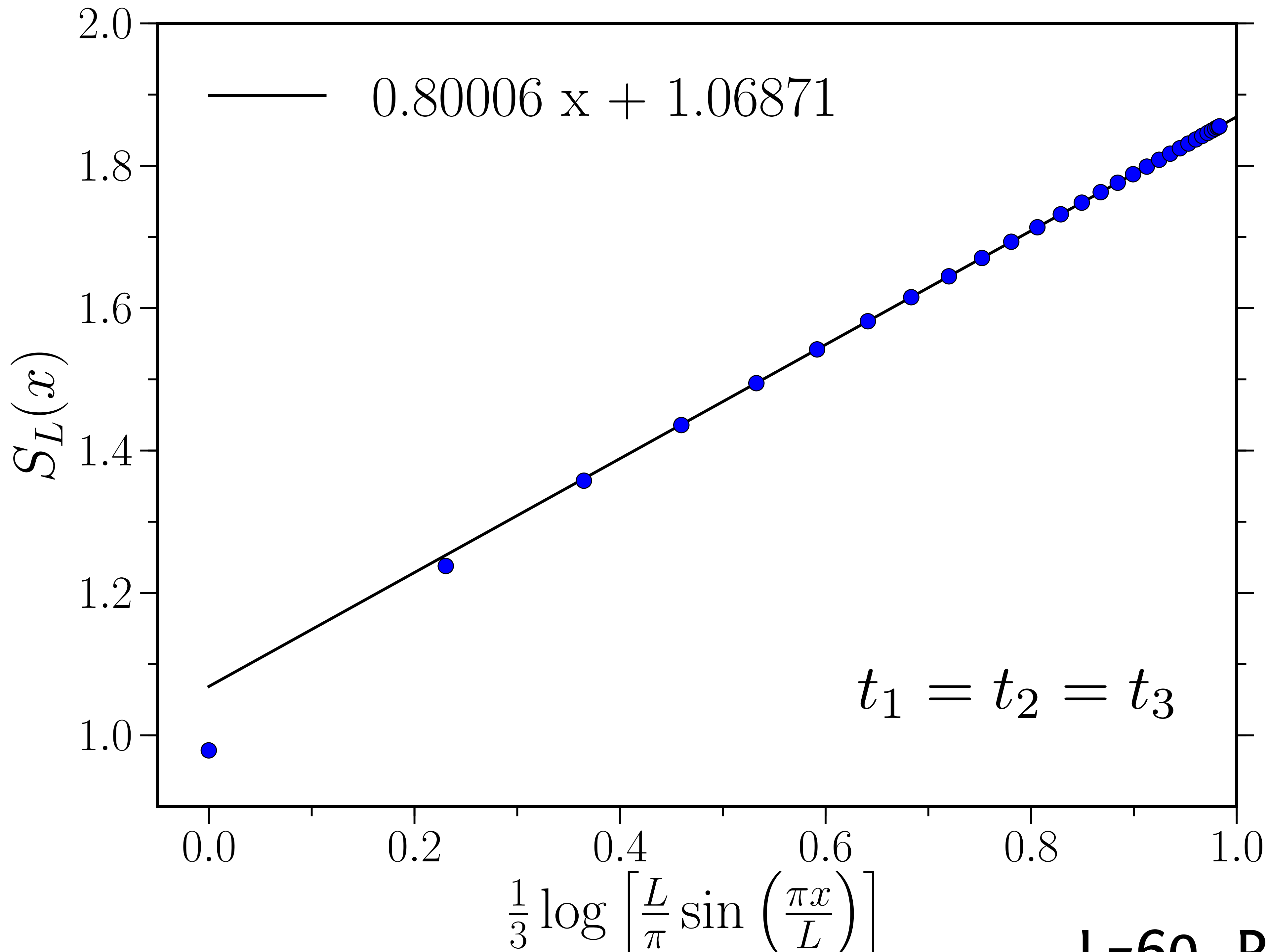


Fine tuning $0 < t_1 = t_2 \ll 1$ couples only left mover of bottom chain to right mover of top chain

Other fields remain gapless,
critical ladder described by *single* Z_3 pfn. field theory

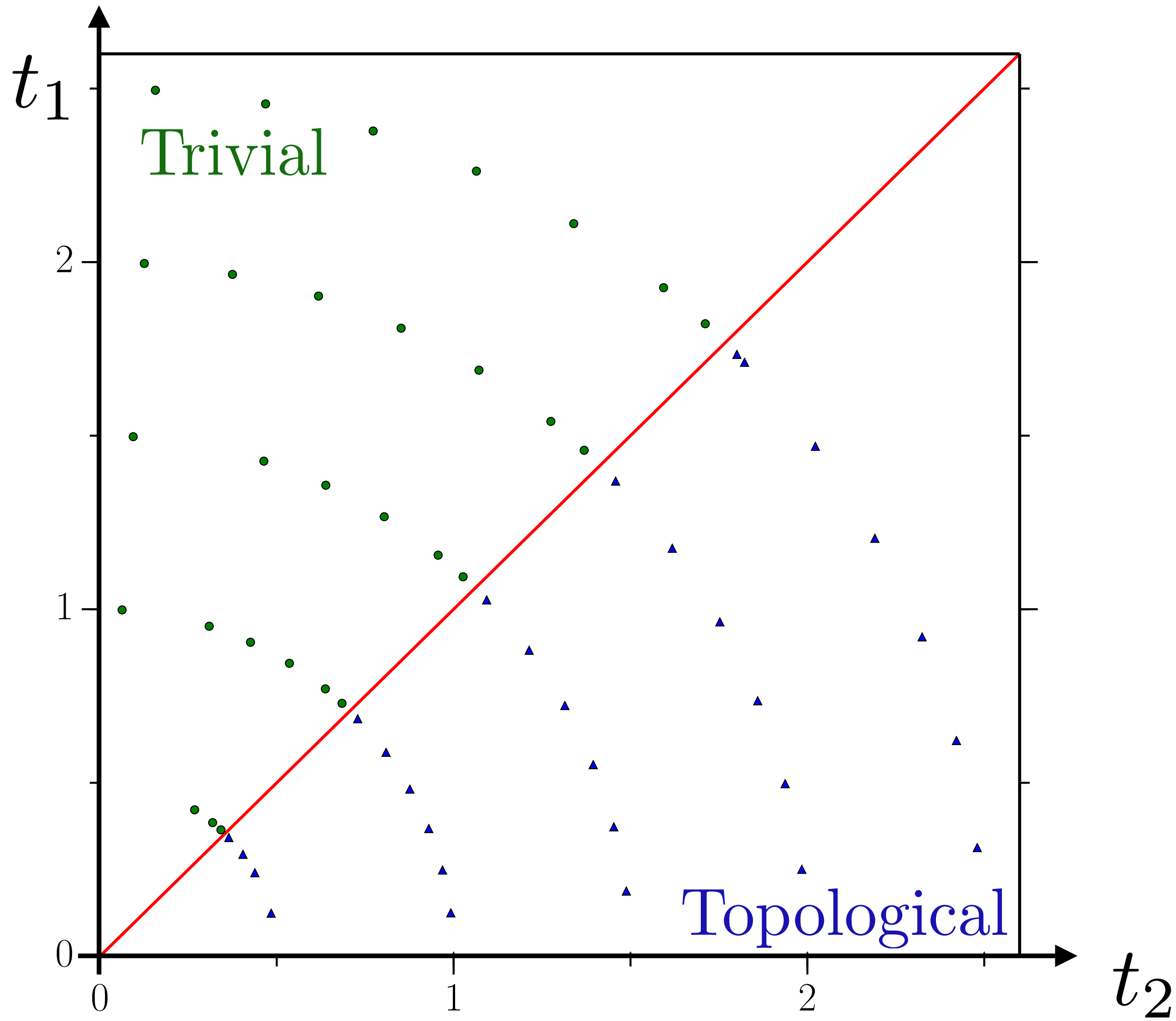


Z_3 parafermion CFT has central charge c=4/5 (=0.8) Confirmed by DMRG on critical ladder

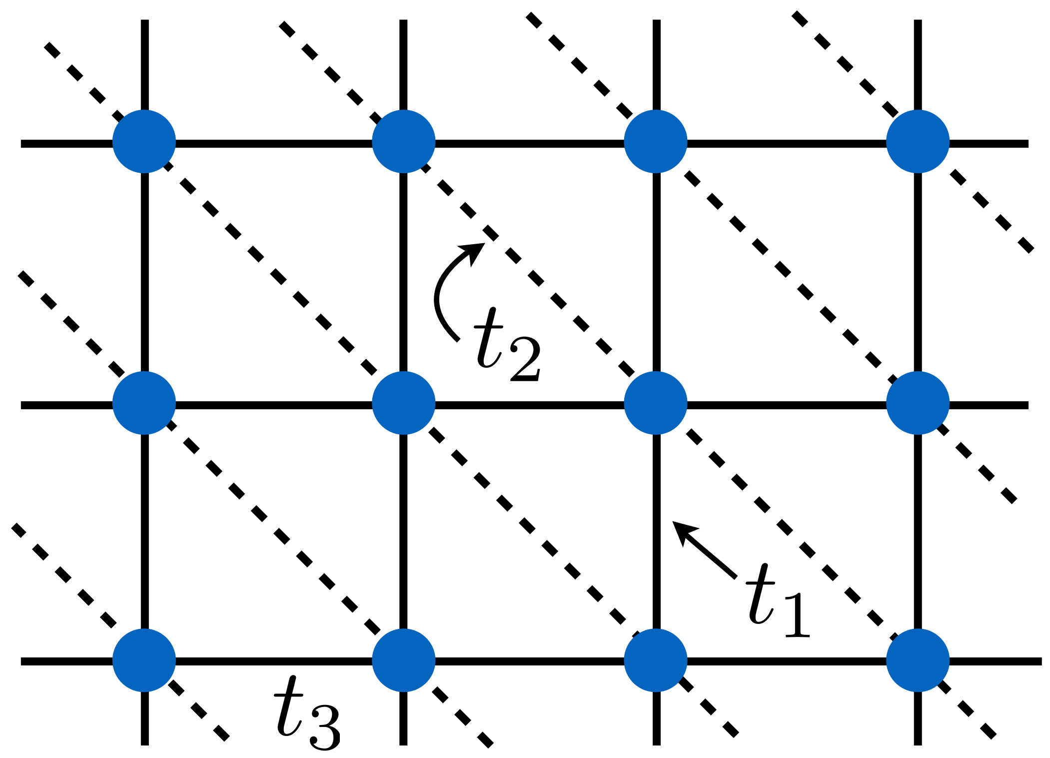


L=60, PBC

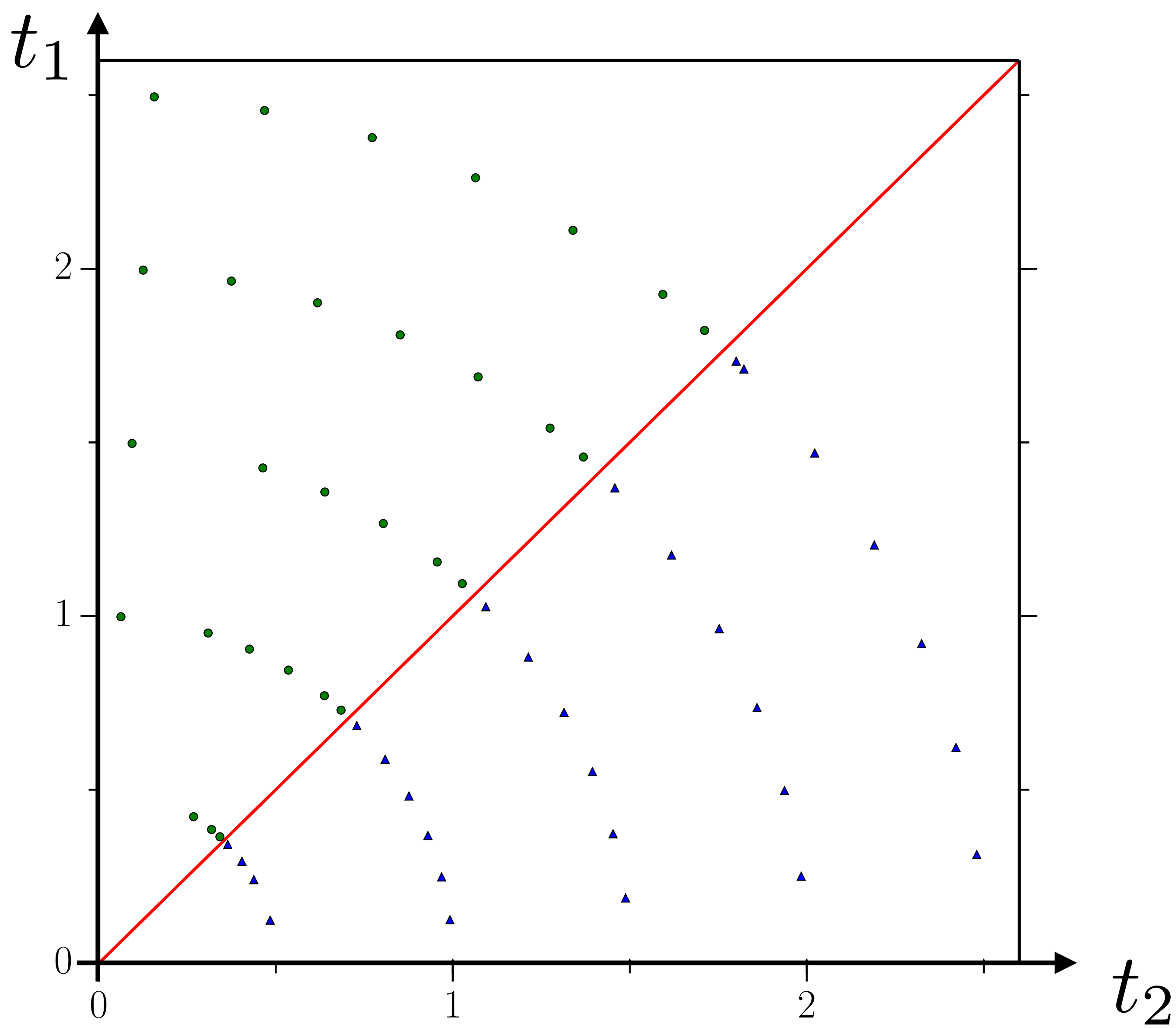
Critical $t_1 = t_2$ line will serve as a precursor of Fibonacci phase in 2d

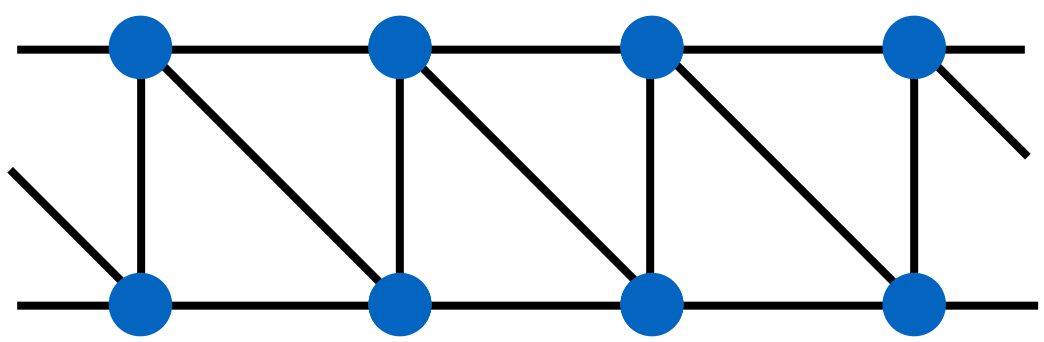


Towards Two Dimensions

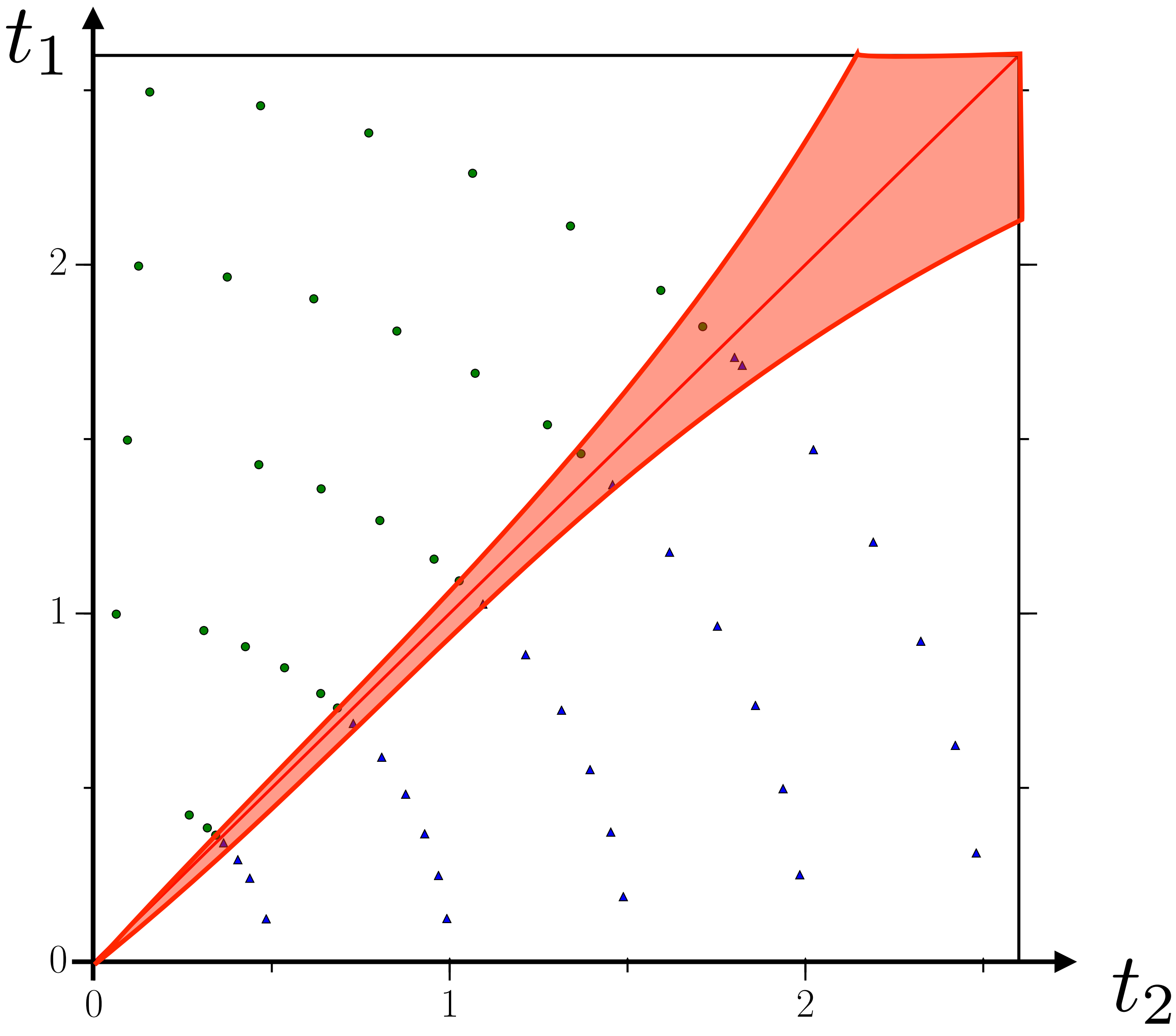


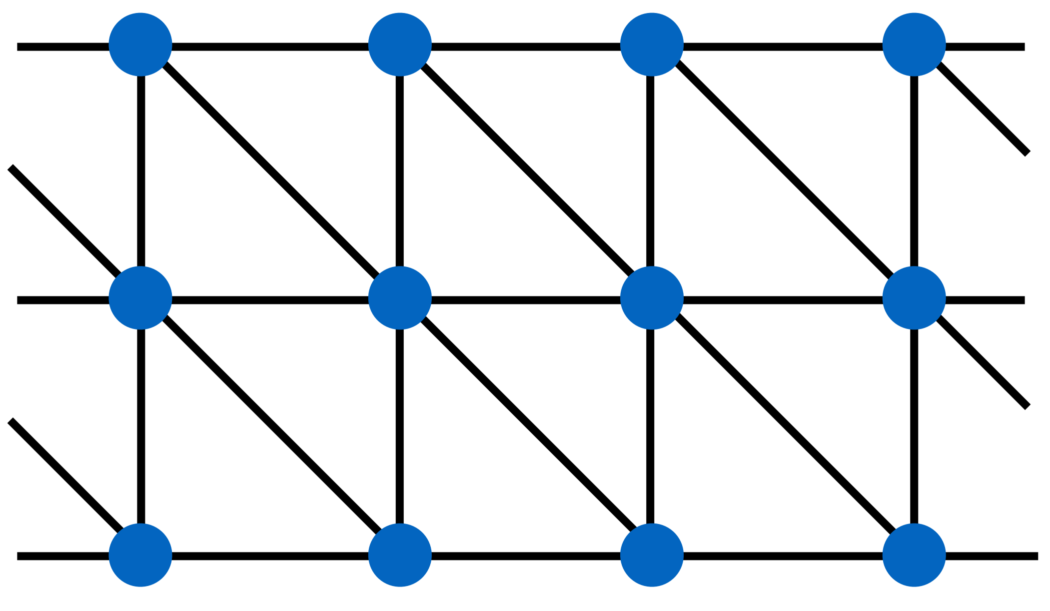
Upon adding more legs,
critical line could become stable 2d phase





Upon adding more legs,
critical line could become stable 2d phase





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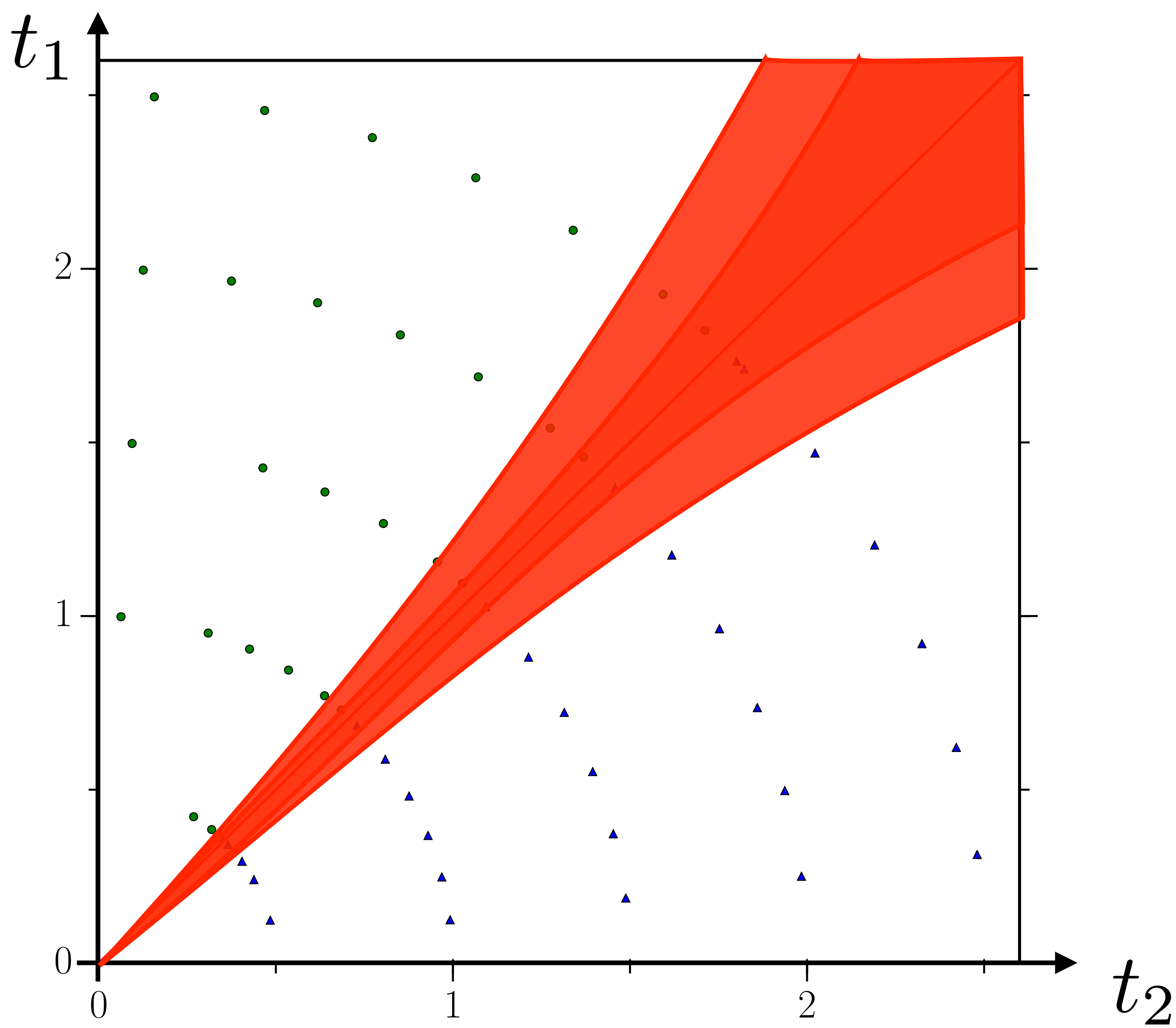
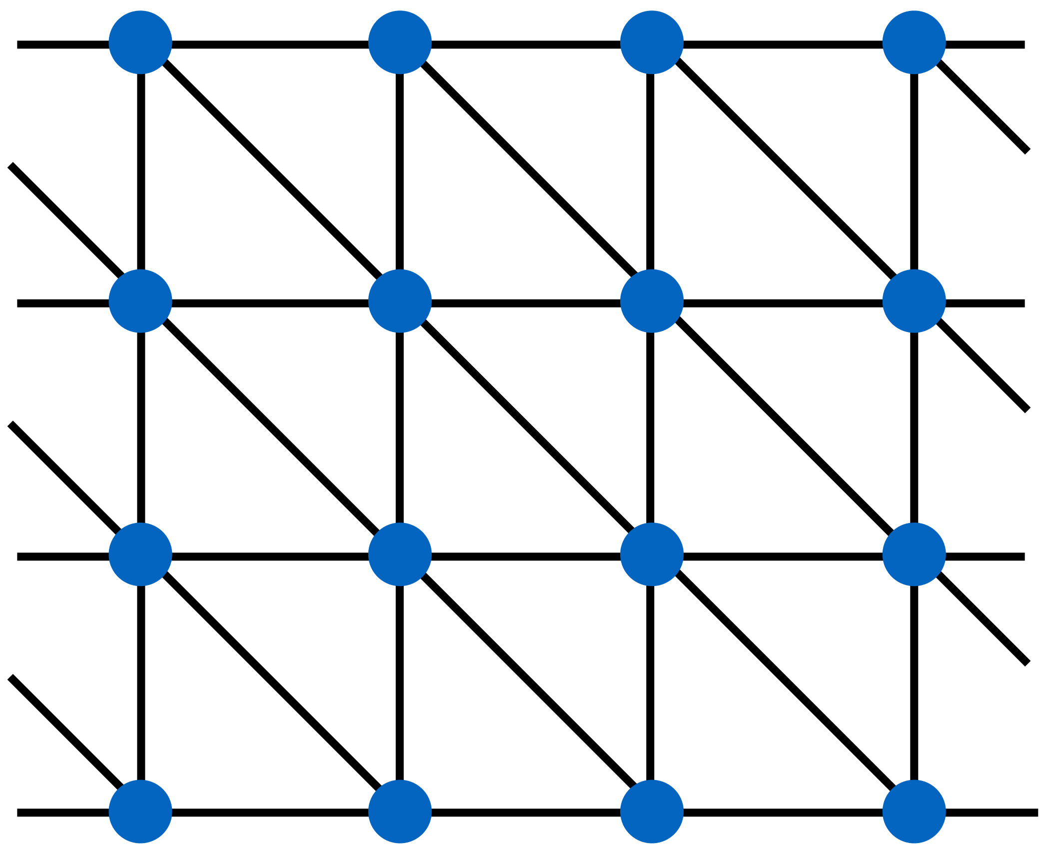
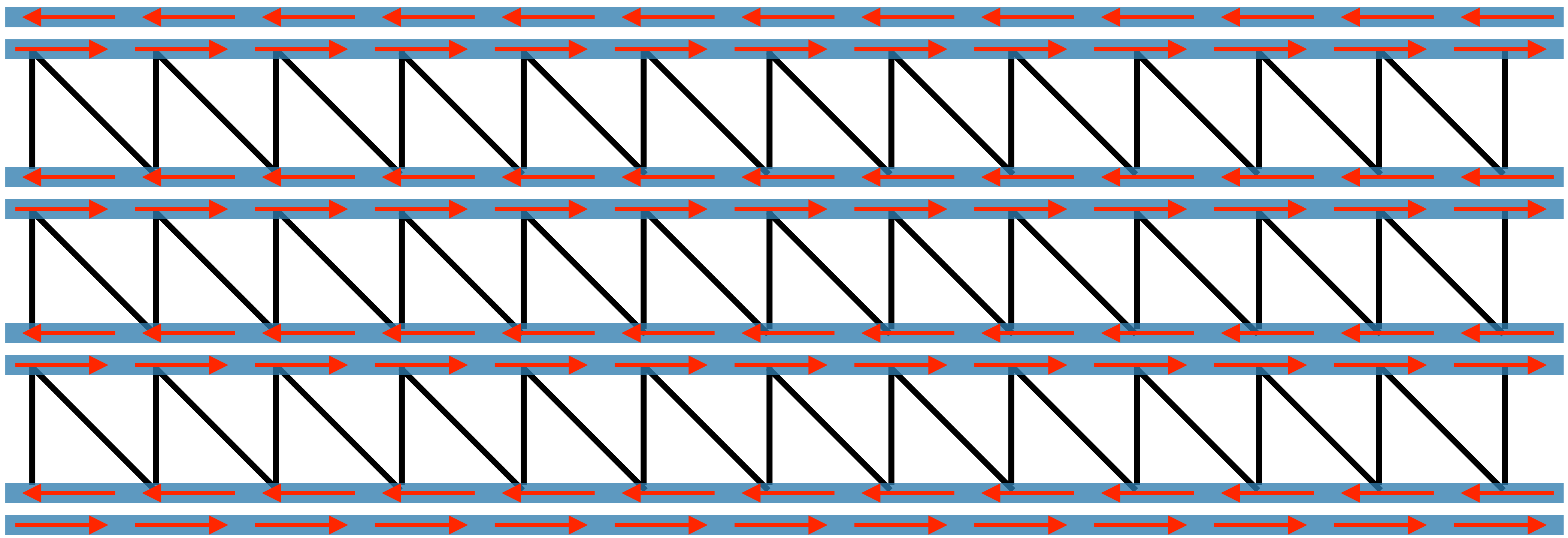




Why expect this?

Iterating field theory argument for weak $t_1 = t_2 > 0$
edge modes get separated by *macroscopic* distance



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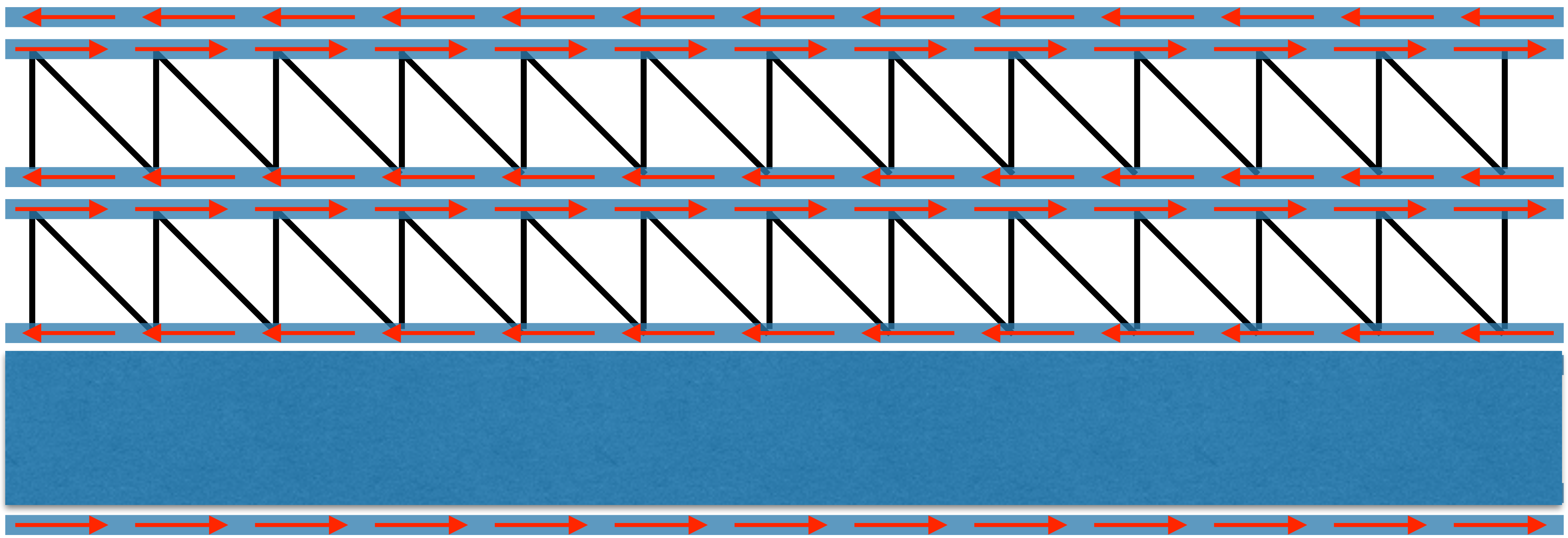

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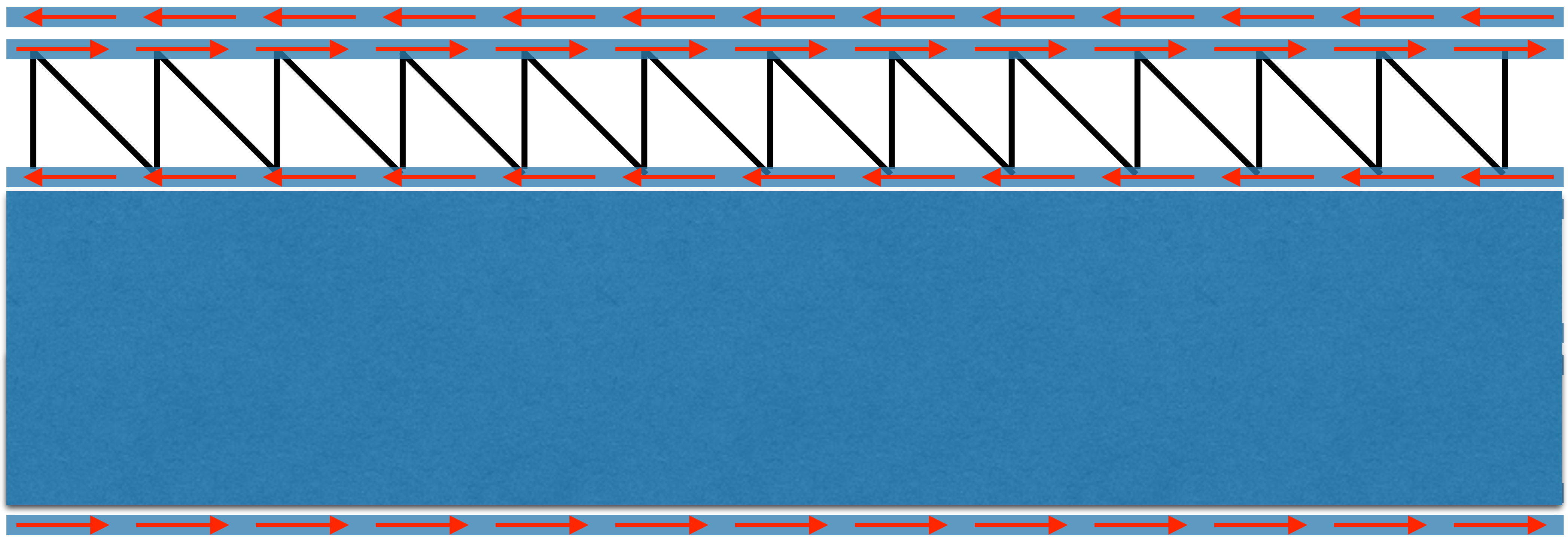

Why expect this?

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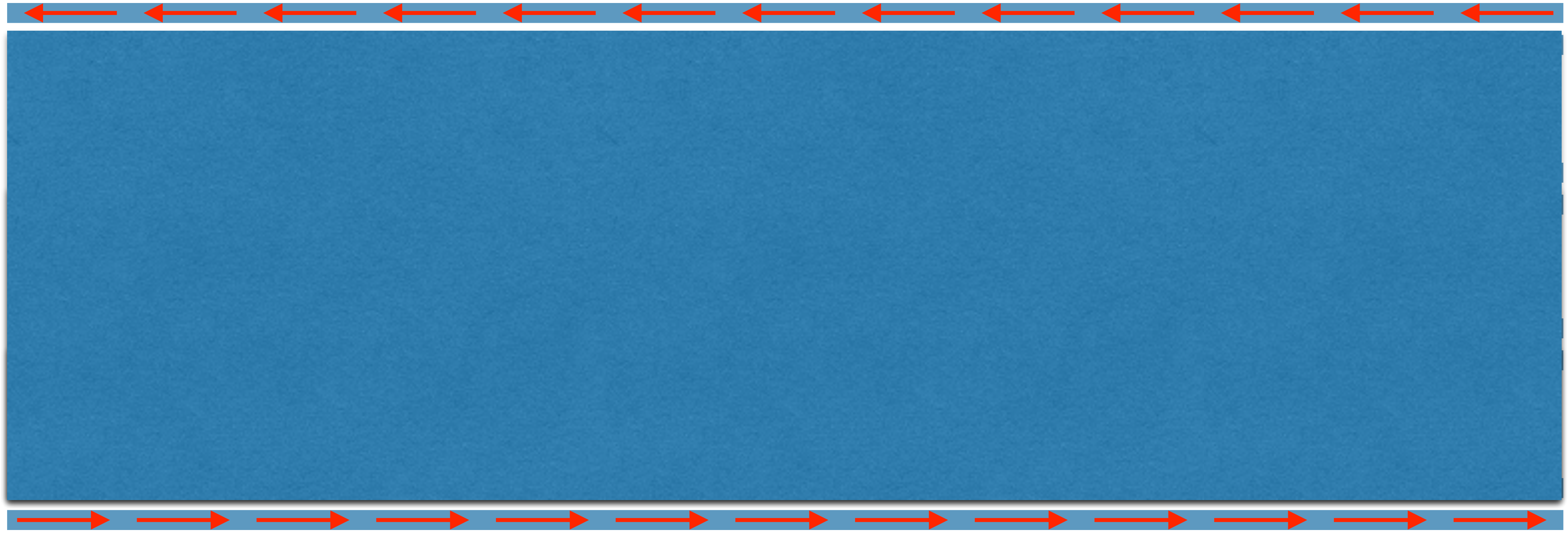
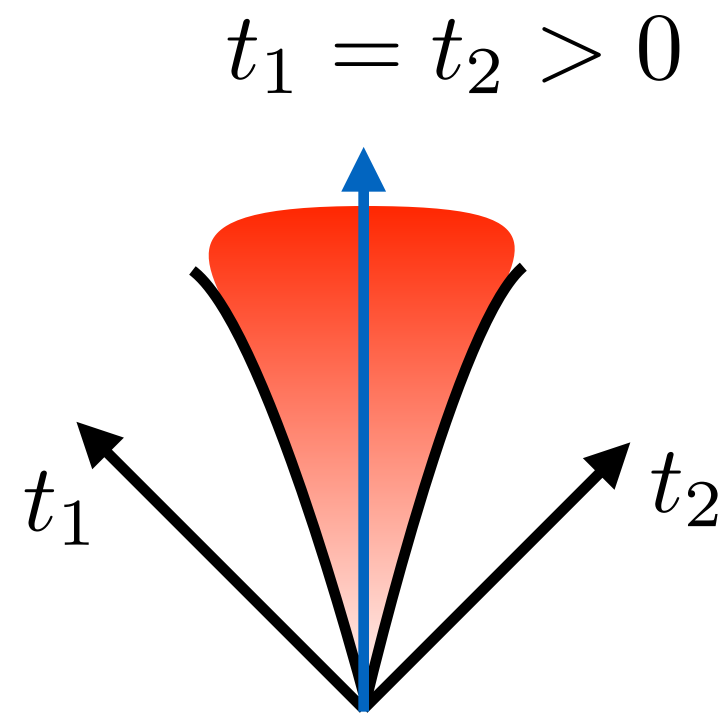
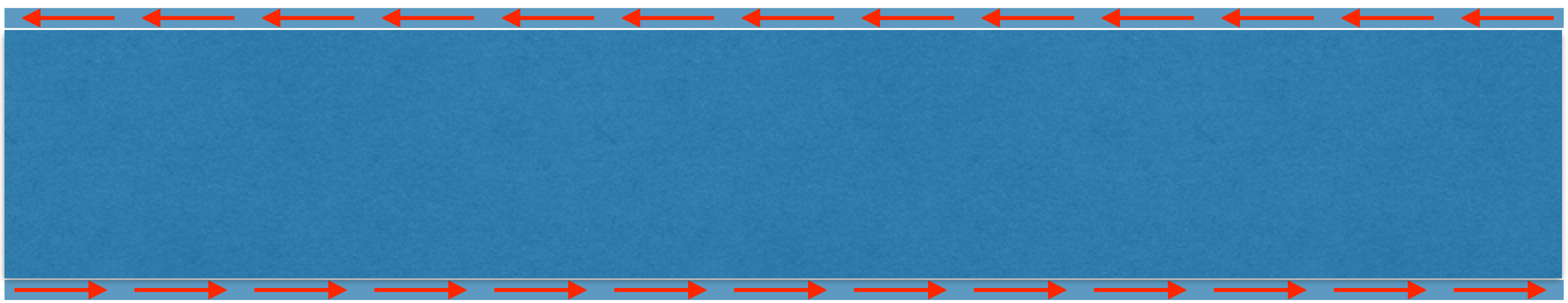


Coupled chain picture thus ‘points’ in interesting direction in parameter space to explore



Presence of chiral gapless edge modes suggests we will reach a *topological phase*



Which one?

Edge theory has six primary fields $\{1, \psi, \psi^\dagger, \sigma, \sigma^\dagger, \epsilon\}$

ψ and $\psi^\dagger$ are continuum limit of lattice parafermions

Treating ψ and $\psi^\dagger$ as local leaves two sectors:

$$\{1, \psi, \psi^\dagger\} \qquad\qquad \{\epsilon, \sigma, \sigma^\dagger\} \ (= \{1, \psi, \psi^\dagger\} \times \epsilon)$$

This implies

$\Longrightarrow$ two degenerate ground states

$\Longrightarrow$ one non-trivial quasiparticle (Fibonacci anyon)

$\Longrightarrow$ counting of low 'energy' entanglement spectra

This phase called the ***Fibonacci phase***

Prior reasoning based on *weakly*-coupled chains

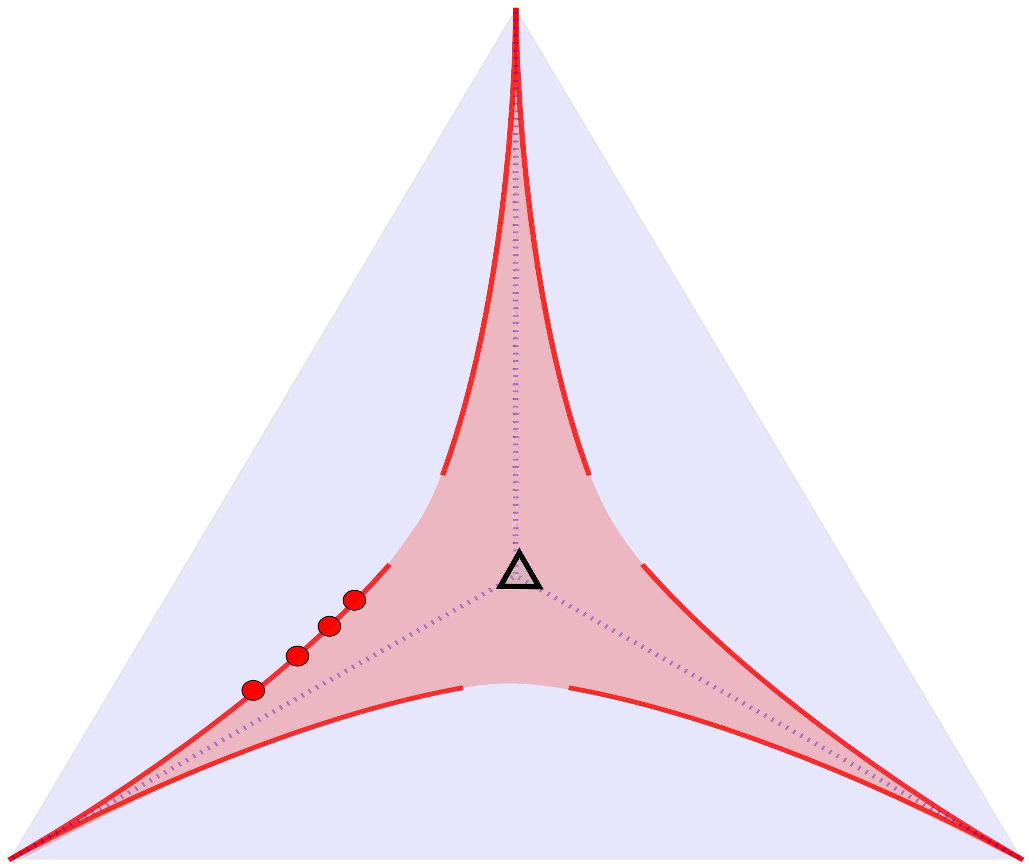
Do subleading interactions eventually couple edge modes?

Stable to finite t_1, t_2 *?*

Does Fibonacci phase persist to isotropic point $t_1 = t_2 = t_3$*?*

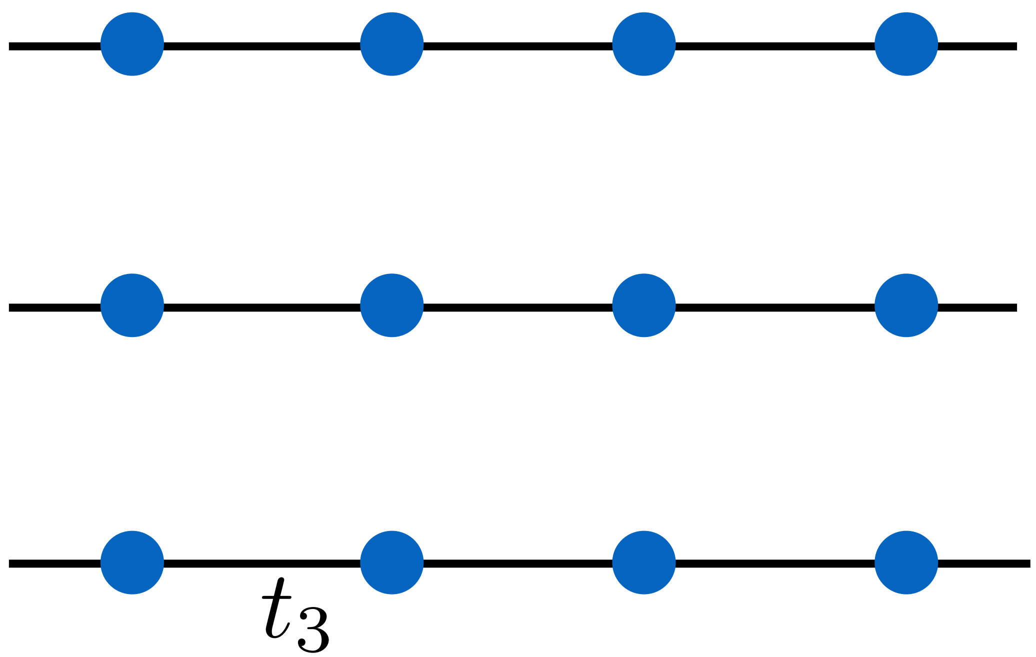
Stability for $t_1 \neq t_2$ *?*

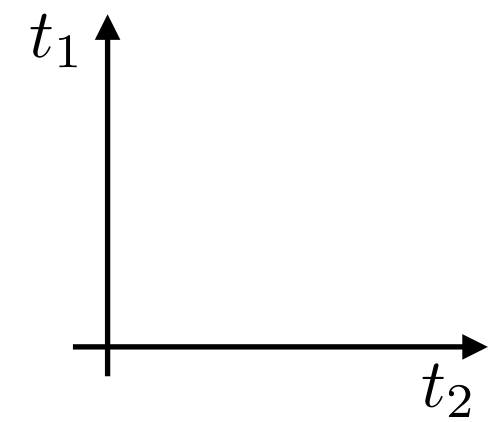
Approach isotropic $t_1 = t_2 = t_3$ limit *non-perturbatively* with DMRG on cylinders



Two-dimensional results: **Fibonacci**

Line of attack

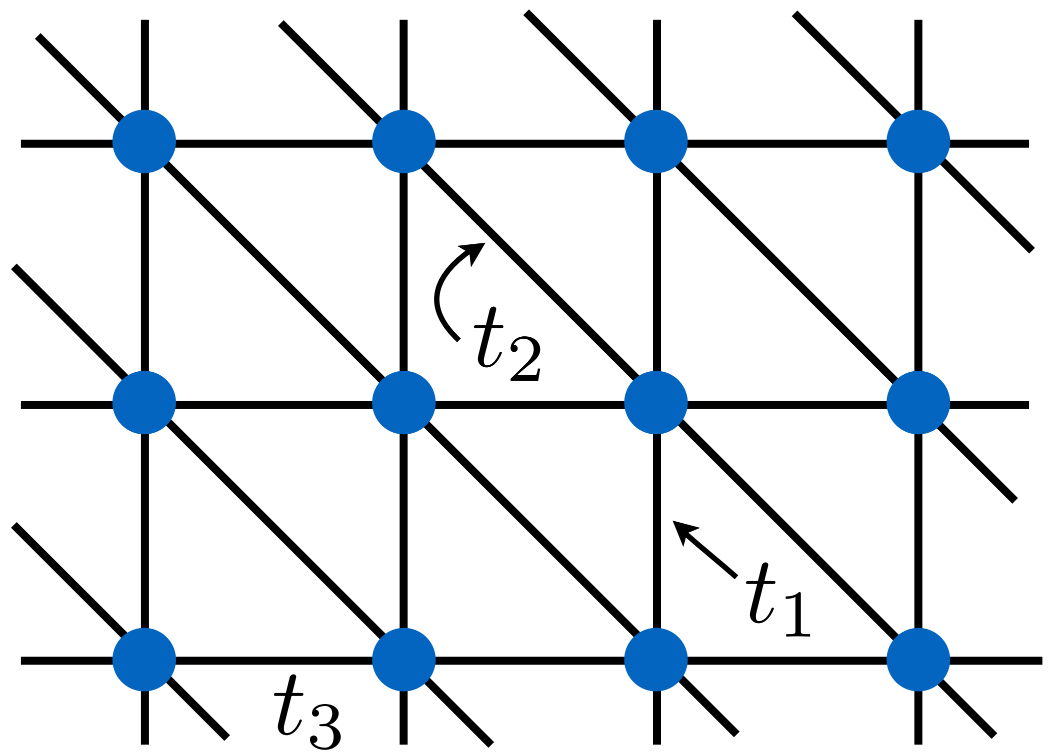


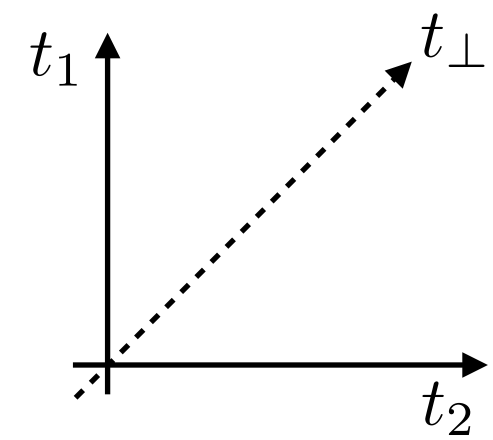


- Gradually increase $t_1 = t_2 \stackrel{\text{def}}{=} t_\perp$ $\quad (t_3 \equiv 1)$
  and number of legs $N_y = 4, 6, 8, 10$

- Apply DMRG to infinitely long cylinders (iDMRG)

Line of attack





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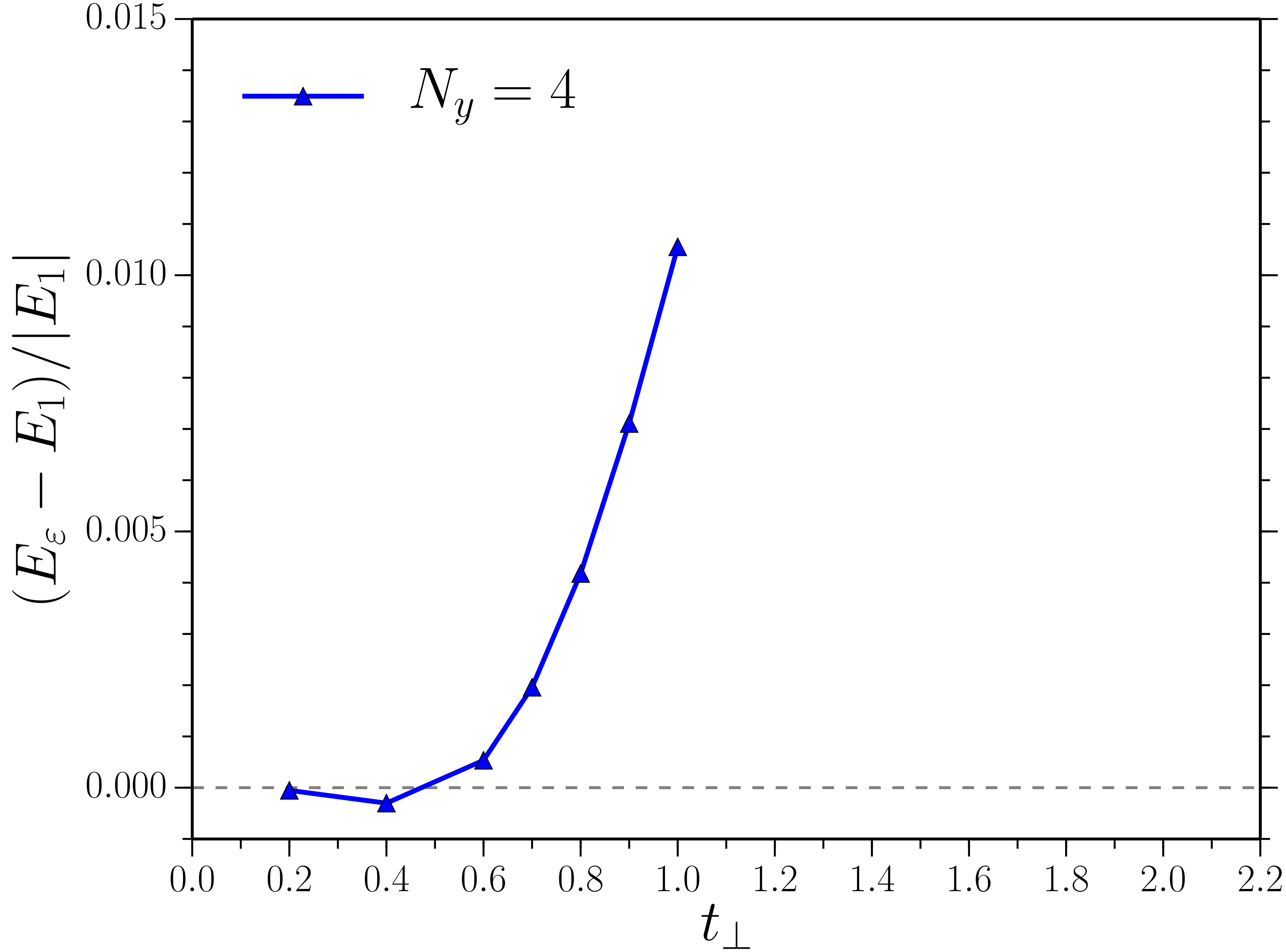
  and number of legs $N_y = 4, 6, 8, 10$

- Apply DMRG to infinitely long cylinders (iDMRG)

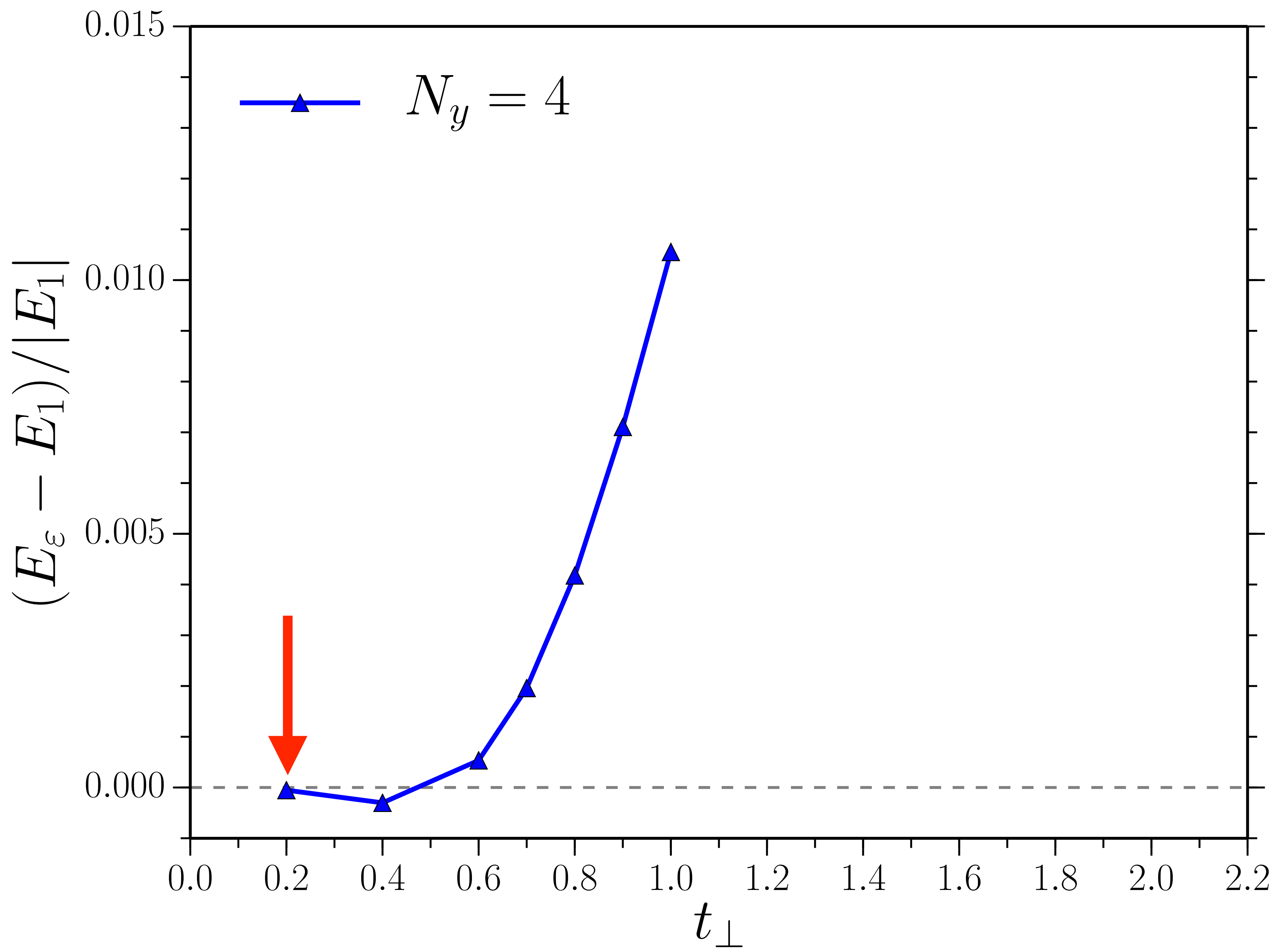
Immediately see two quasi-degenerate ground states

Energy splitting of ground states versus $t_\perp$ for $N_y = 4$:



For small $t_\perp = 0.2$, y- correlation length apparently less than circumference of $N_y = 4$ cylinder

Seeing two-dimensional topological states?



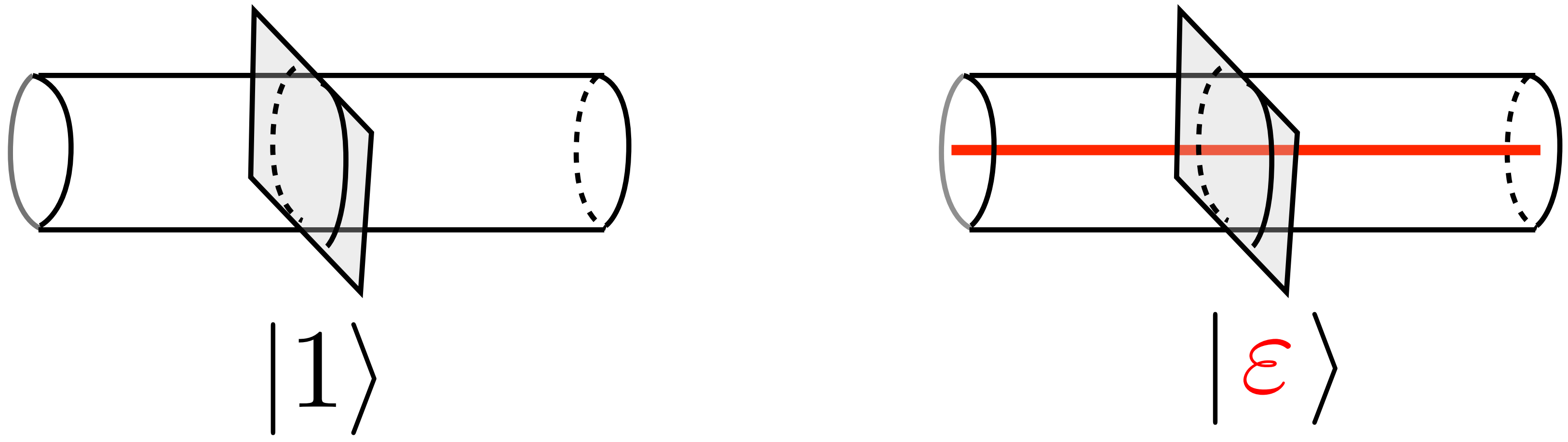
Q:

How to observe physics with no local order parameter?

How to distinguish degenerate ground states?

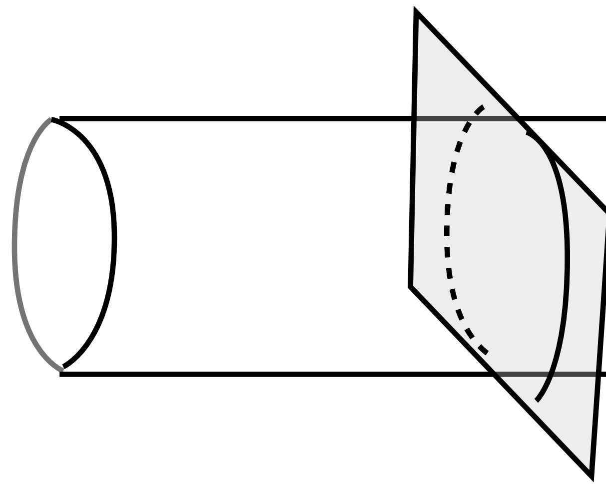
A:

Entanglement entropy and entanglement ‘spectrum’
by ‘cutting’ the wavefunction



“Entanglement spectrum” is set of probabilities for system to be in different states near the cut

$$|\Psi\rangle =$$



$$= \sum_n$$

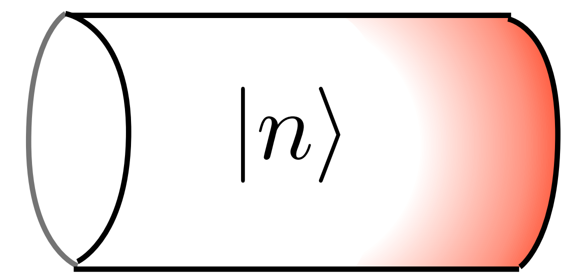
$$\times\; p_n \qquad \left(p_n \stackrel{\mathrm{def}}{=} e^{-\tilde{E}_n} \right)$$

“Entanglement entropy” measures log(# states) system fluctuates through

$$S = -\sum_n p_n \log p_n$$

Entanglement spectra of ground states show sharp degeneracies

$$|1, \varepsilon\rangle = \sum_n$$

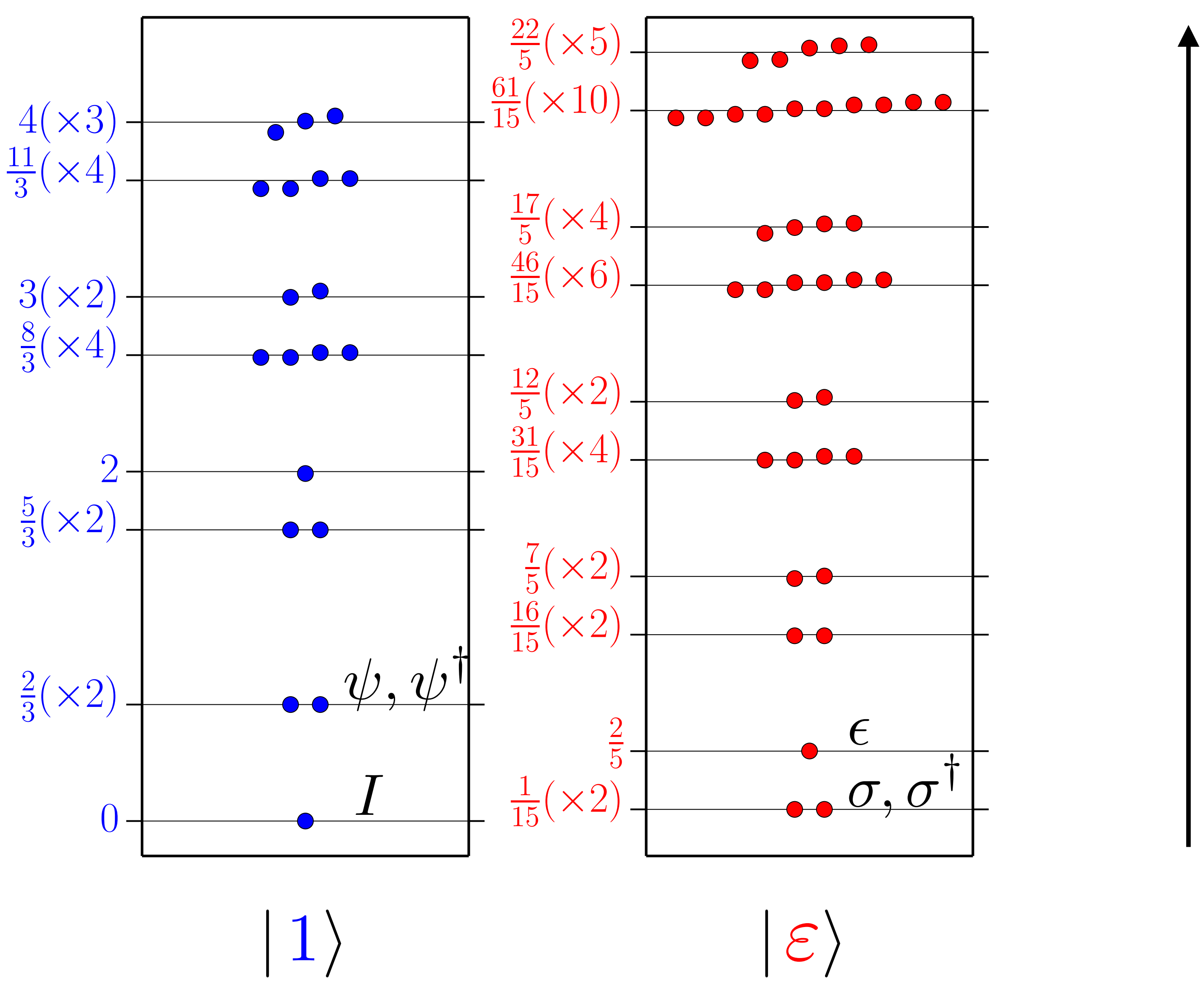


$$e^{-\tilde{E}_n}$$

Spectrum of “virtual edge” has precise agreement with field theory (Z3 parafermion CFT) of edge spectrum

$t_\perp = 0.2$

$N_y = 4$



$\tilde{E}_n = -\log p_n$

(shifted and rescaled)

From finite-size scaling, can measure topological entanglement entropy

Prediction for these topological states[1,2,3]

$$S_1 = aN_y - \gamma_1$$

$$S_\varepsilon = aN_y - \gamma_\varepsilon$$

$$\gamma_1 = \log(\mathcal{D}) \simeq 0.6430$$

$$\gamma_\varepsilon = \log(\mathcal{D}/\phi) \simeq 0.1617$$

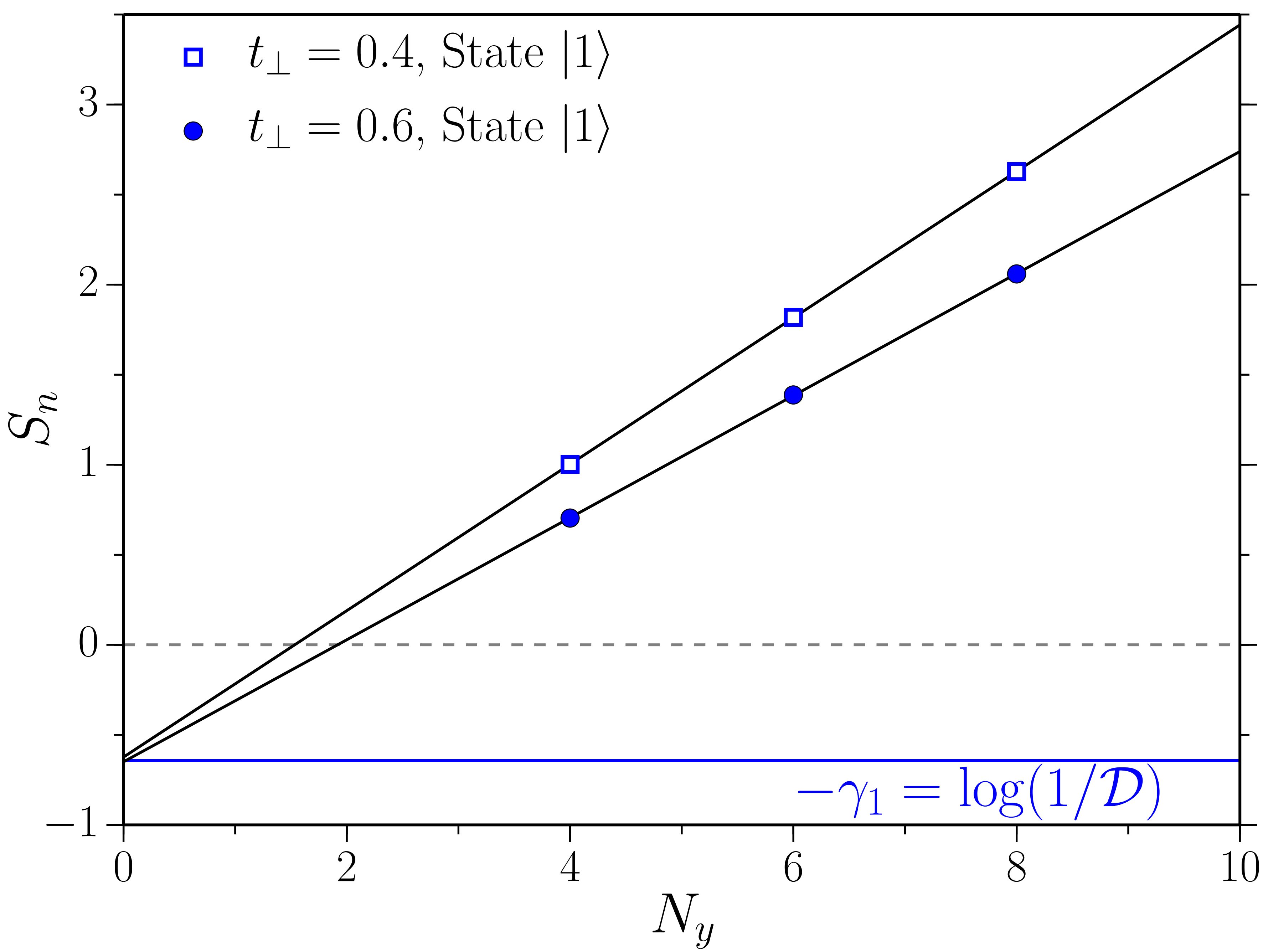
Constrained quantum fluctuations

$$\mathcal{D} = \sqrt{1+\phi^2}$$

$$\phi = (1+\sqrt{5})/2$$

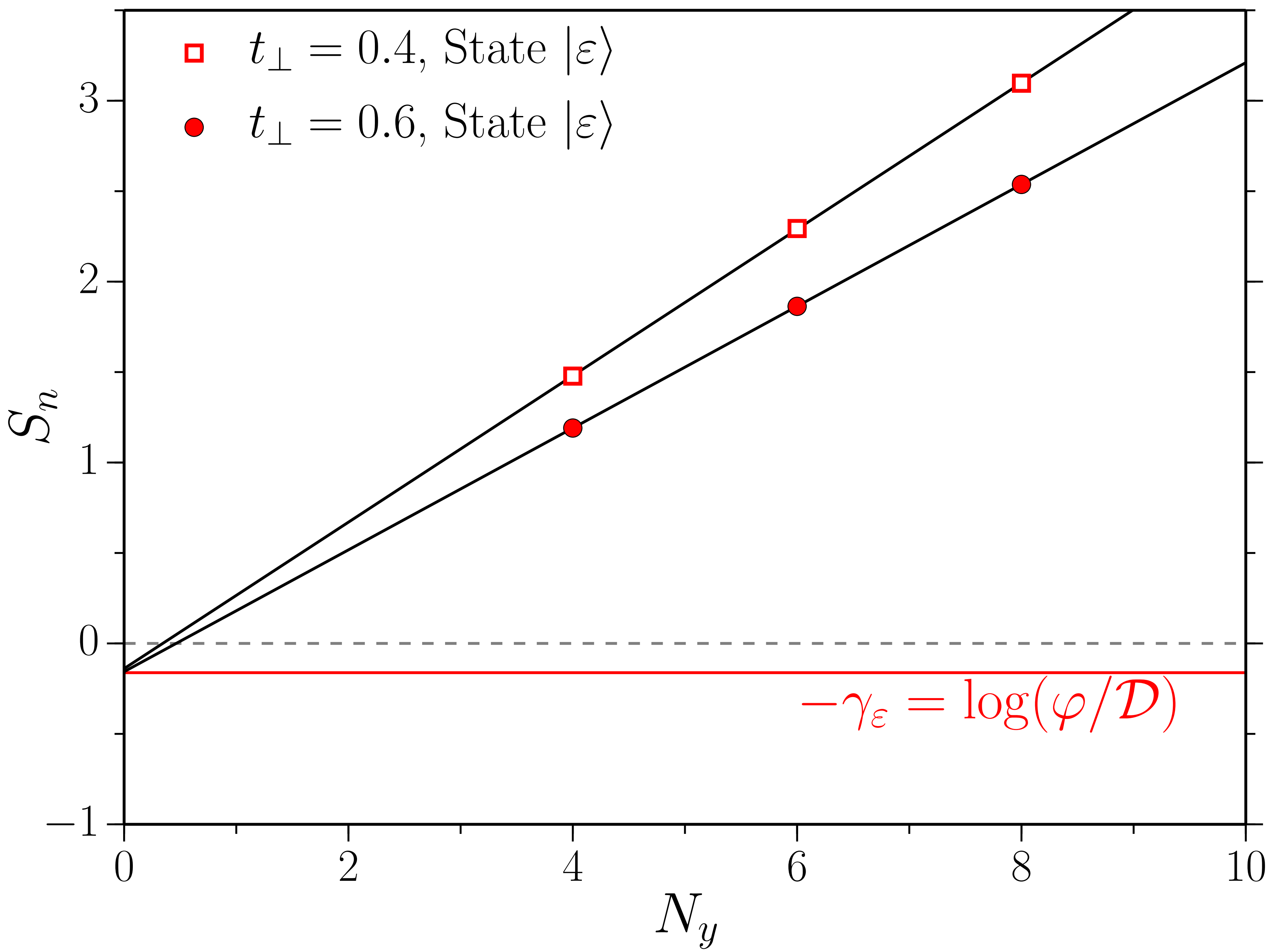
1) Levin, Wen PRL 96, 110405 (2006)
2) Kitaev, Preskill PRL 96, 110404 (2006)
3) Zhang, Grover, Turner, Oshikawa, Vishwanath, PRB 85, 235151 (2012)

Topological entanglement entropy, state $|1\rangle$ (two strengths of $t_\perp$)*



* Up to $-\log\sqrt{3}$ shift

Topological entanglement entropy, state $|\varepsilon\rangle$ (two strengths of $t_\perp$)*



* Up to $-\log\sqrt{3}$ shift

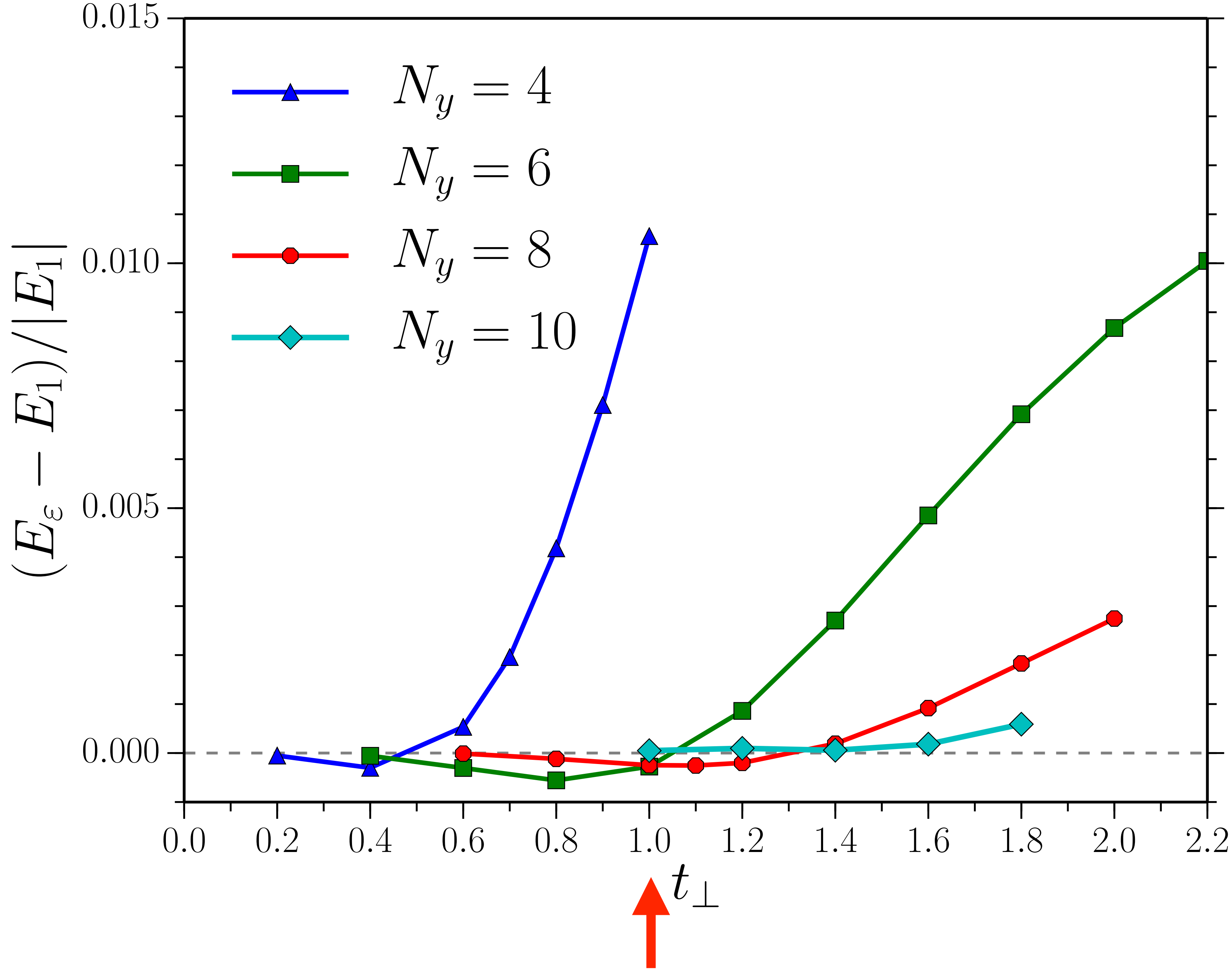
Topological entanglement entropy shows completeness of ground states

	γ_1	γ_ε	$e^{-2\gamma_1} + e^{-2\gamma_\varepsilon}$
Exact	$\log \mathcal{D} \approx 0.6430$	$\log(\mathcal{D}/\varphi) \approx 0.1617$	1
$t_\perp = 0.4$[a]	0.6235	0.1393	1.0442
$t_\perp = 0.4$[b]	0.6306	0.1538	1.0186
$t_\perp = 0.6$	0.6498	0.1562	1.0043

All ground states accounted for

[a] N_y=4,6,8 fitted
[b] Only N_y=4,6, fitted

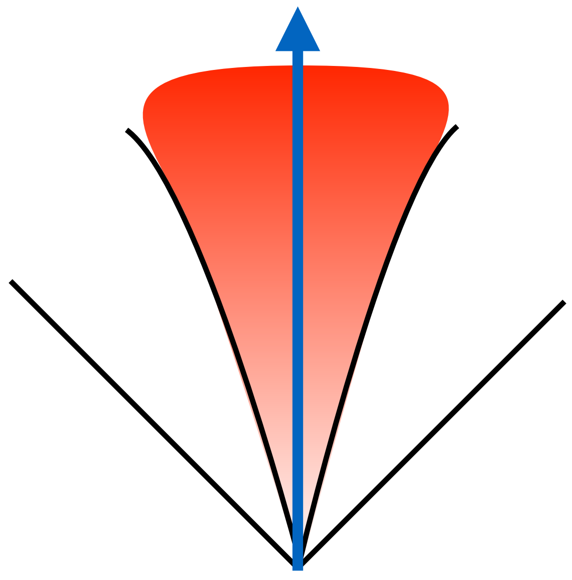
Approach isotropic limit on larger cylinders, energy splitting:



Fibonacci phase at isotropic triangular lattice and beyond

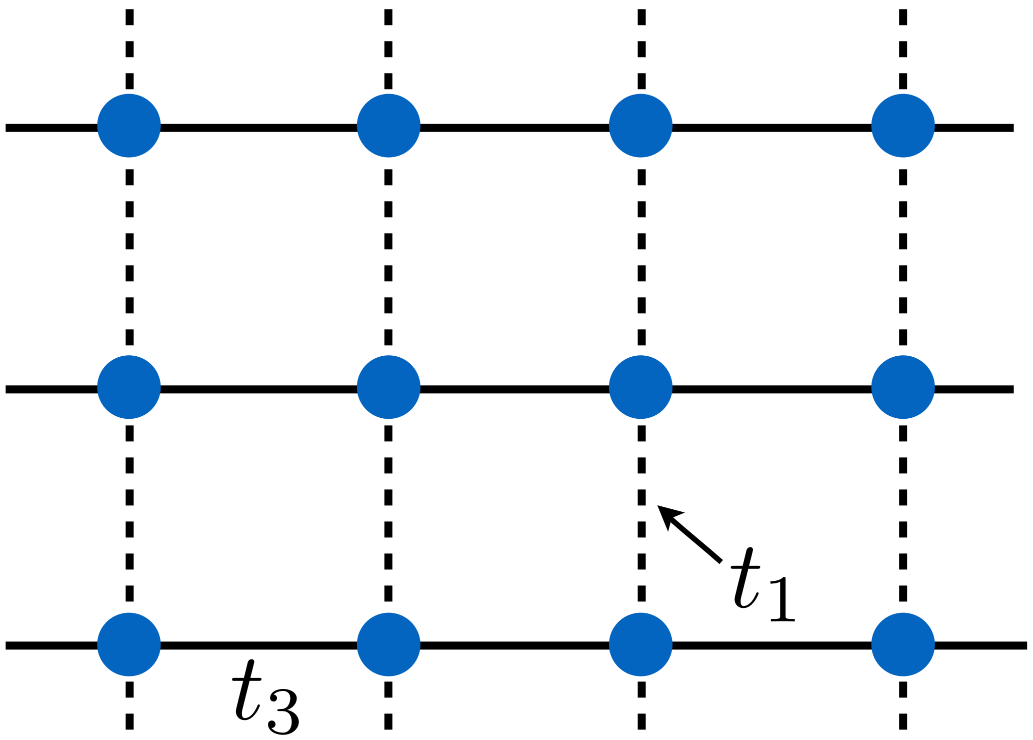
Strong evidence that isotropic triangular lattice of Z_3 parafermions lies deep within Fibonacci phase

Weakly-coupled wires approach safely guided us deep into gapped, topological phase



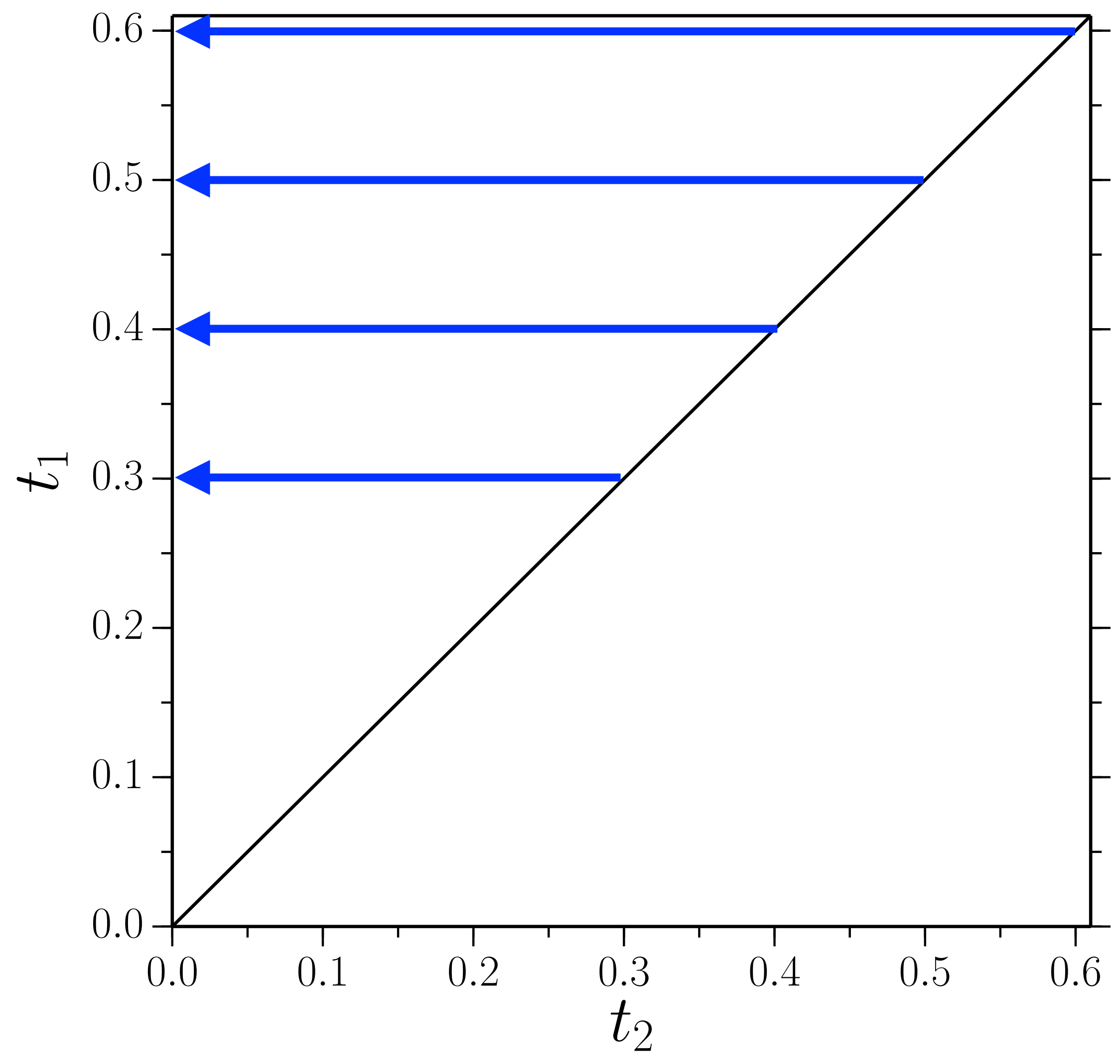
Initial results for anisotropic square lattice yield no evidence of Fibonacci phase

Different phase?



Adiabatically move toward square lattice

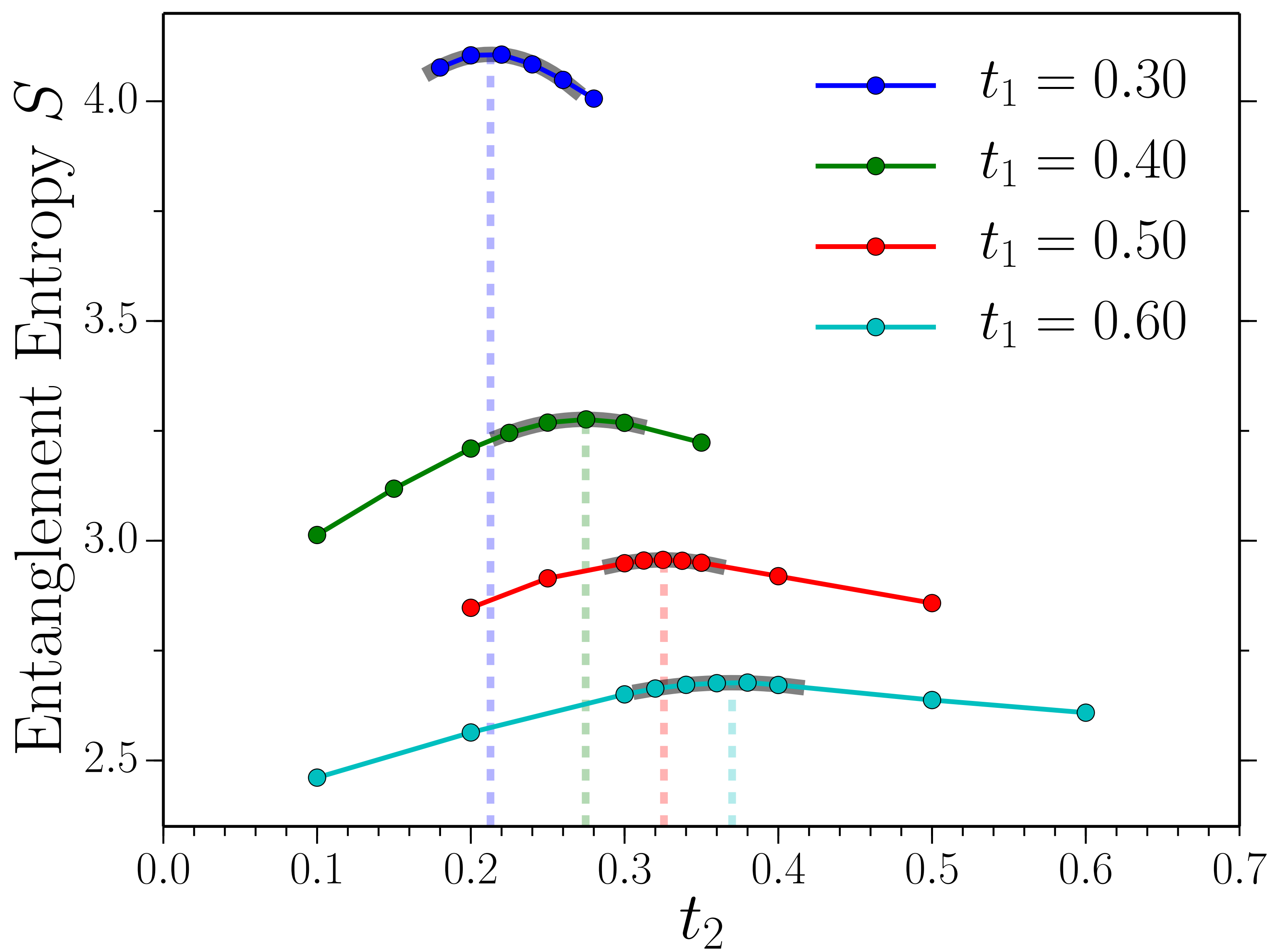
- fix value of t_1
- gradually reduce t_2 to zero



Measure entanglement entropy along these lines

Observe peaks in entanglement entropy

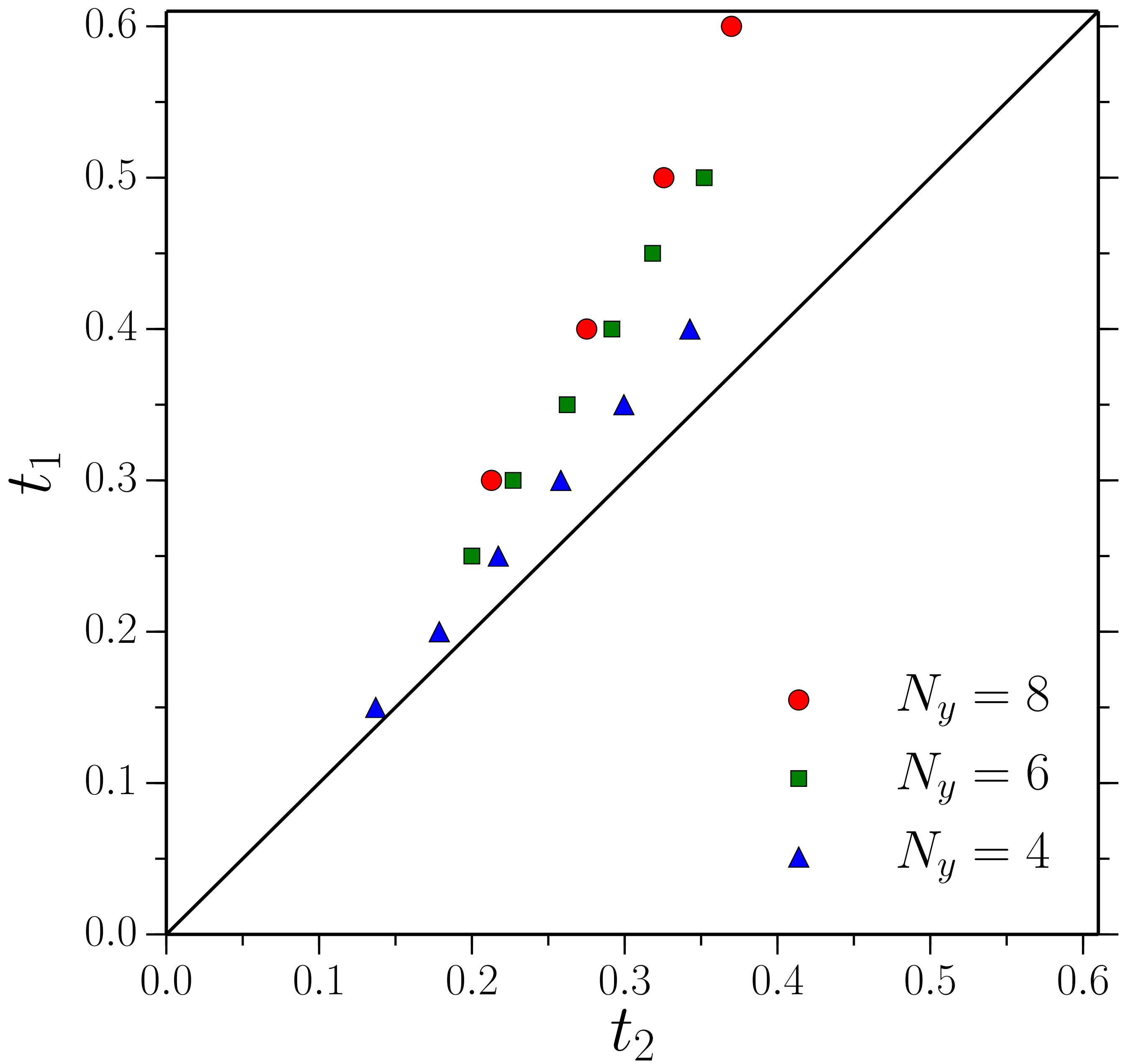
Results for $N_y = 8$ (largest)



Empirically fit peaks to quadratic to estimate location

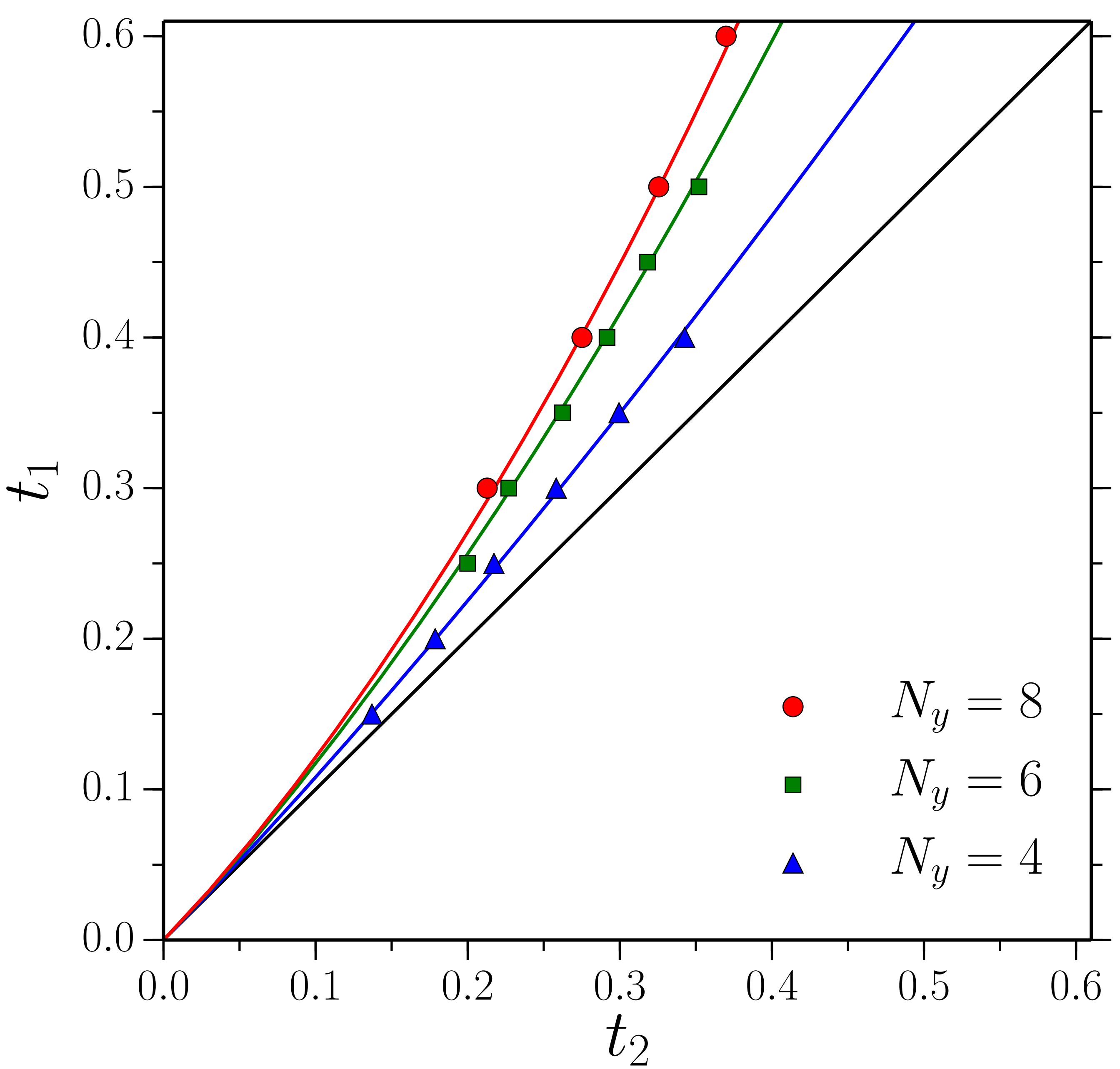
Combining results for $N_y = 4, 6, 8$

Peak locations:

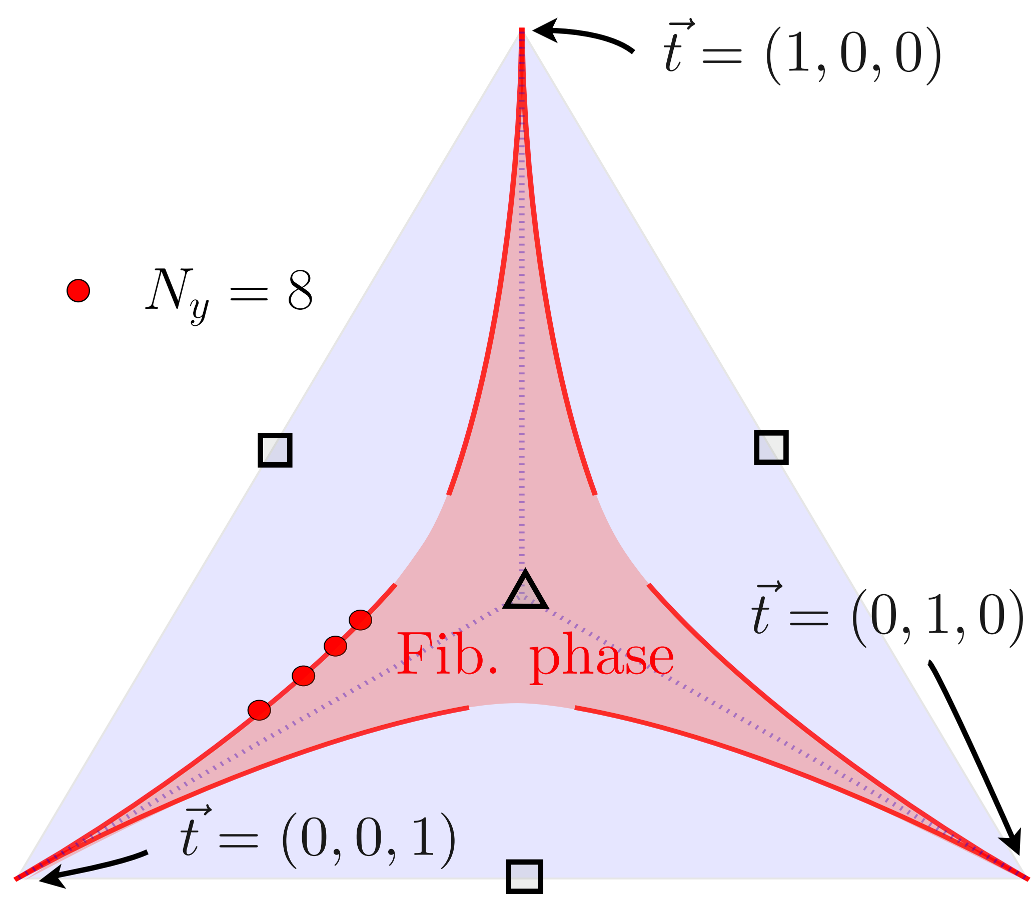


RG argument predicts critical line along

$$t_{2c} - t_{1c} = C(t_{2c} + t_{1c})^{8/5}$$



Transforming $N_y = 8$ fit under all permutations of t_1, t_2, t_3 gives estimate for phase boundary



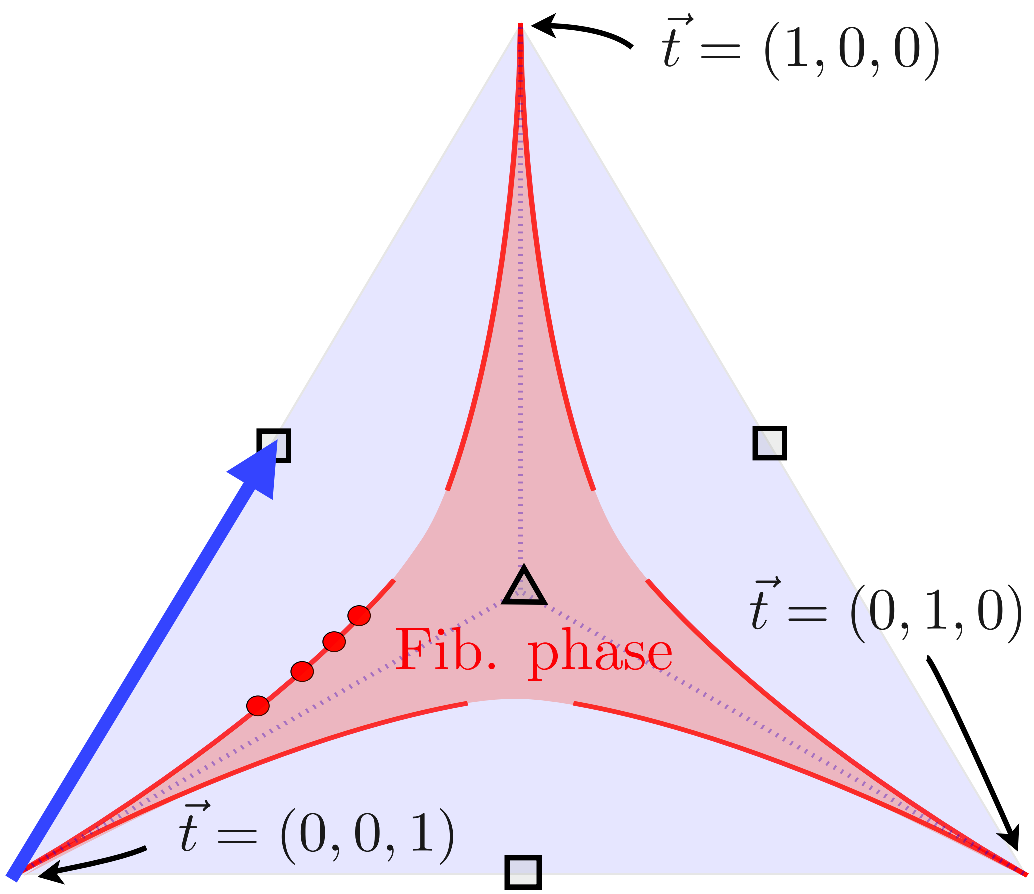
$\vec{t} = (t_1, t_2, t_3)$

△ Isotropic triangular point

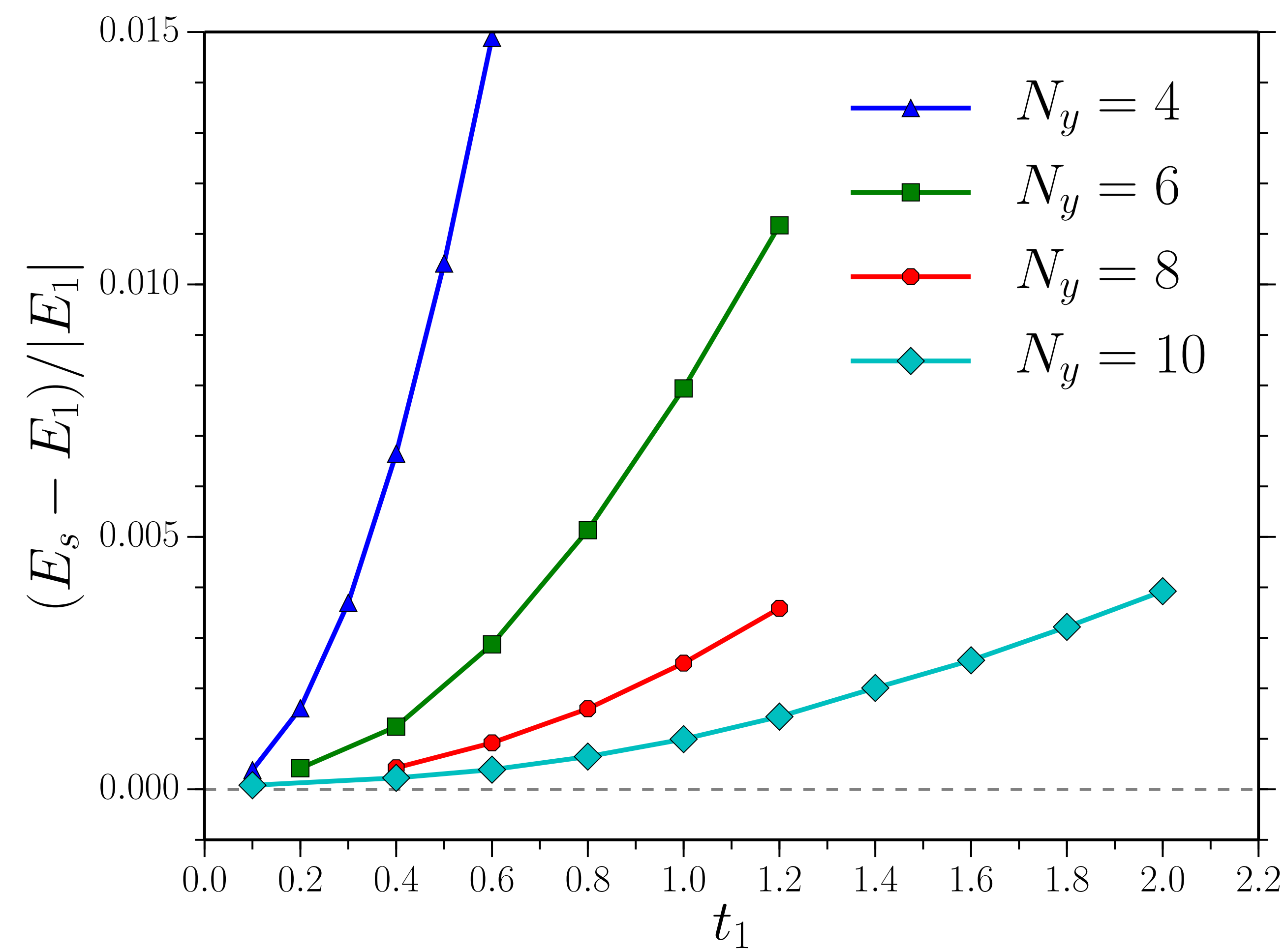
□ Isotropic square point

Square lattice in different phase, but direct attack not useful

Two degen. ground states, but large finite-size effects



Energy splitting:

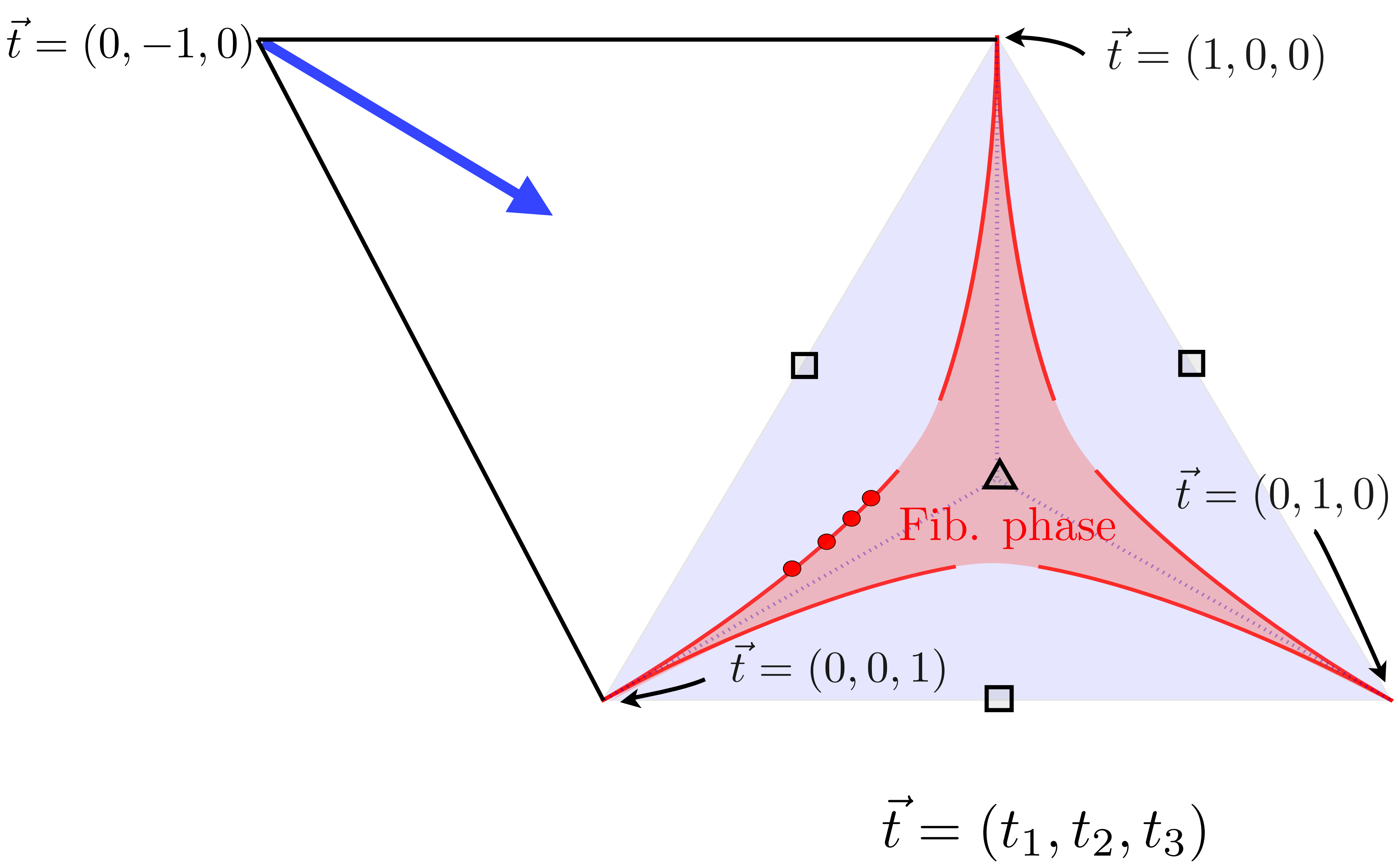


$\vec{t} = (t_1, t_2, t_3)$

△ Isotropic triangular point

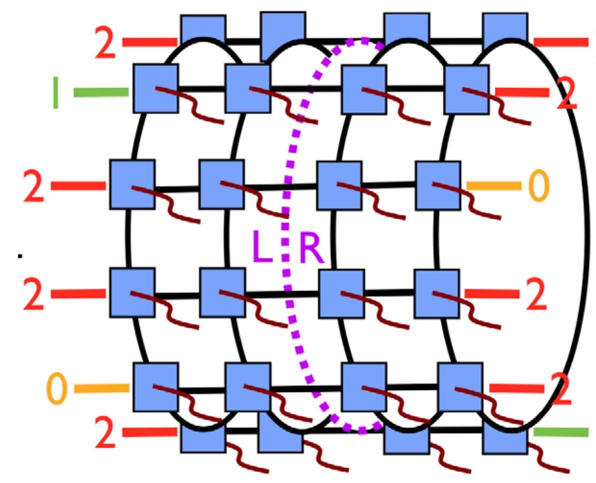
□ Isotropic square point

So we attacked from limit of decoupled chains with negative-sign of interactions



Similar coupled chain argument + DMRG numerics finds topological phase but no Fibonacci anyon

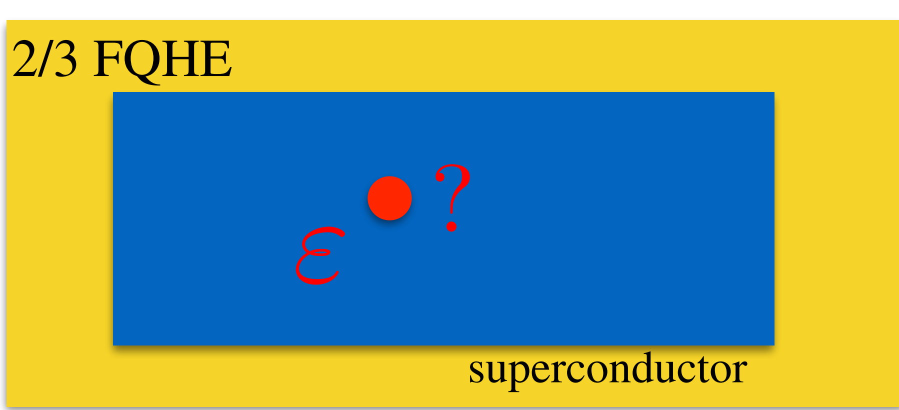
Wrap up

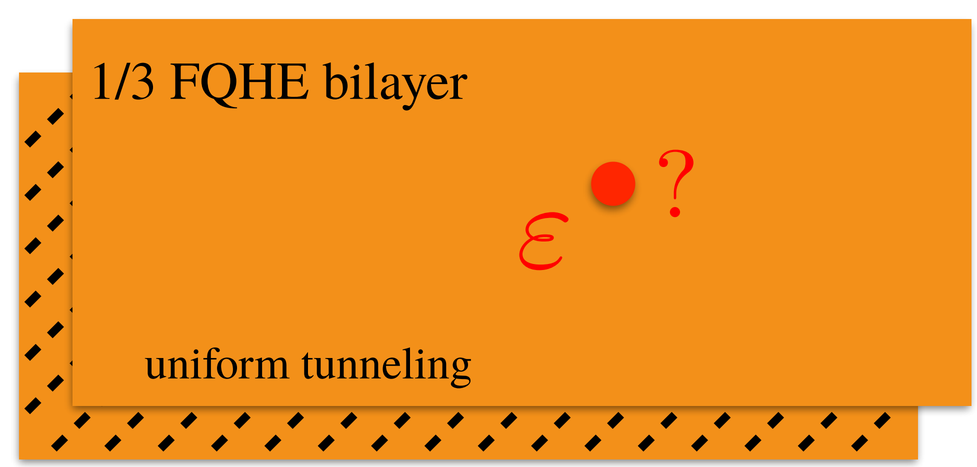


In this talk, showed that an isotropic, next-neighbor model of coupled parafermions realizes a highly non-trivial 2D phase (*Fibonacci phase*)

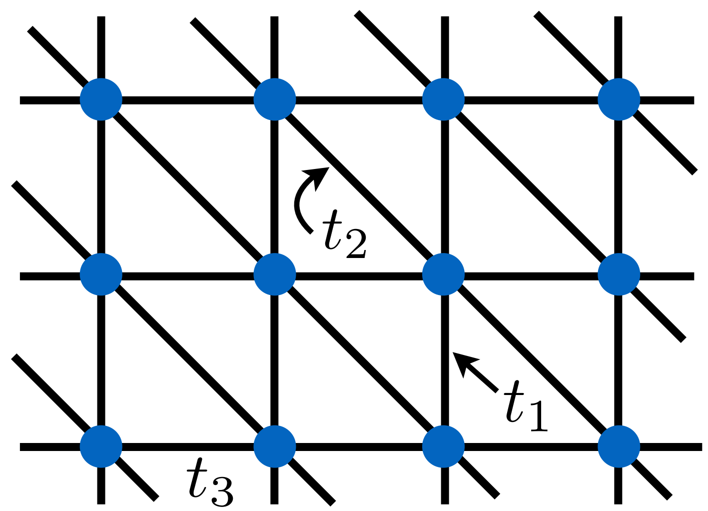
Could guide search for 'smeared out' limit of such a model, for example

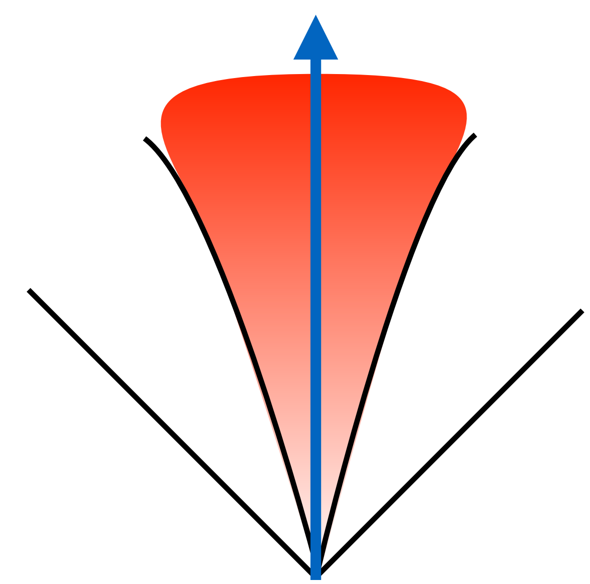
- uniform superconductor coupled to 2/3 fractional QHE
- coupled fractional QHE bilayers





More generally,
weakly-coupled chain analytics
+ DMRG style numerics
= fruitful approach for discovering simple lattice models deep in interesting phases

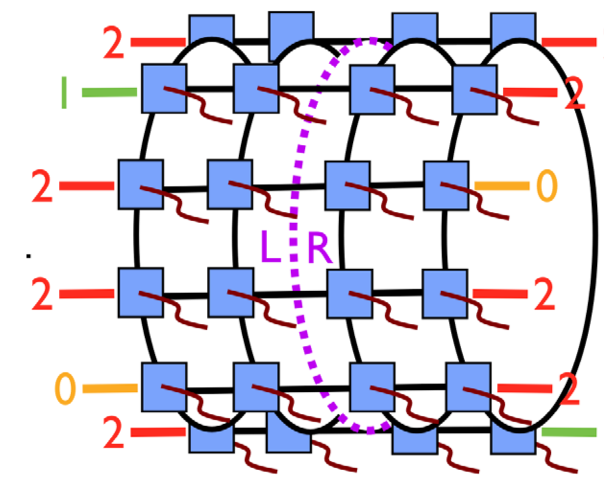




Other short-range lattice models for topological phases?

Useful for finding 2D phases without gapless edges?

"Beyond DMRG" methods are coming



Poilblanc, J. Stat. Mech. P10026 (20

Efficient schemes for contracting / optimizing infinite 2D variational wavefunctions (so called Tensor Product States / PEPS)[1,2]

Known how to write topological states as simple tensor product states...

Study proximate phases by adding small number of variational parameters

1) Evenbly, Vidal 1412.0732 (2014)
2) Lubasch, Cirac, Banuls, PRB 90, 064425 (2014)

Summary

- Isotropic triangular lattice of (Z_3) parafermions lies deep within Fibonacci phase
- Isotropic square lattice likely hosts a different (Abelian) topological phase
- Powerful combination of coupled-chain analytics + DMRG numerics