Quantum quenches in 2D via arrays of coupled chains

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Quantum quenches in 2D via arrays of coupled chains

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- Introduction
  - many body states with MPS
  - entanglement and dimension
- MPS for arrays of coupled chains
- Time evolution for arrays of coupled chains
  - Perturbative limit
  - ‘Dynamical phase transitions’

Representing Many-Body Quantum States

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General many-body state

$$|\psi\rangle = \sum_{\{\sigma\}} c_{\sigma_1 \cdots \sigma_N} |\sigma_1 \cdots \sigma_N\rangle$$

Need exponentially many c numbers to completely specify state

$$d_\sigma^N = \dim(\sigma)^N$$

But typical states we deal with need far fewer

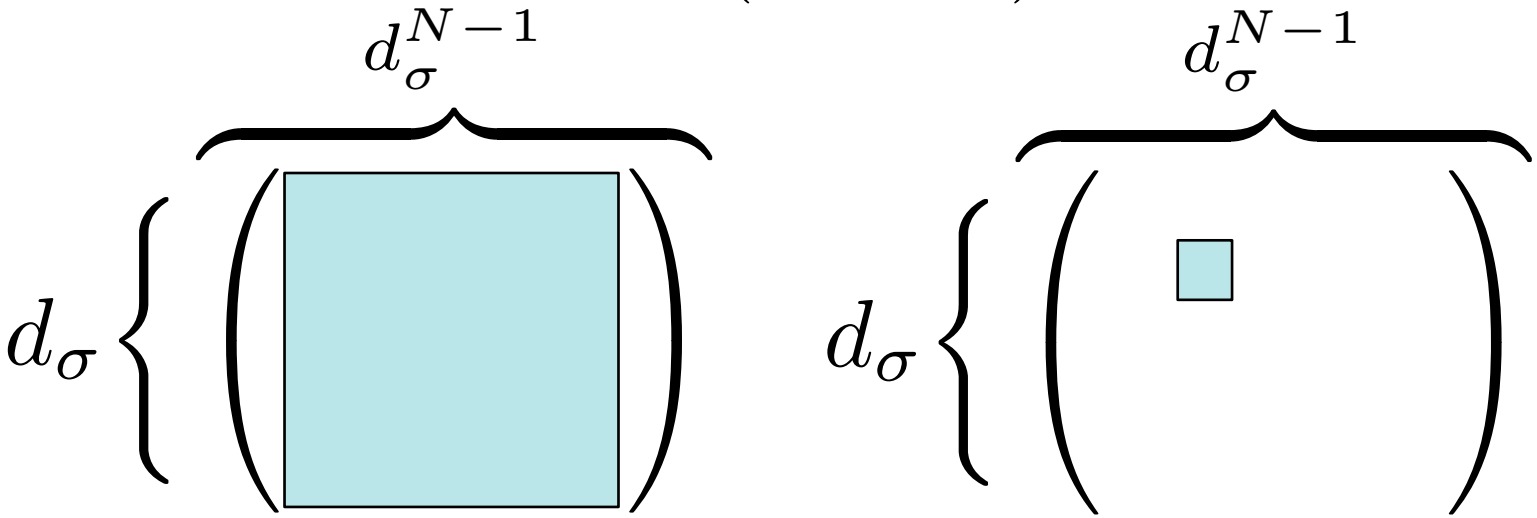
In particular for a simple product state only need 1

$$|\psi\rangle = |\sigma_1 \cdots \sigma_N\rangle$$

What is an efficient representation between these extremes?

How do we choose which contributions are ‘small’ and can be neglected?

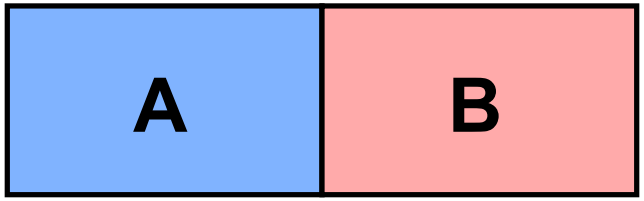
$$c_{\sigma_1,(\sigma_2\cdots\sigma_N)}$$



General

Product state

A bipartite system



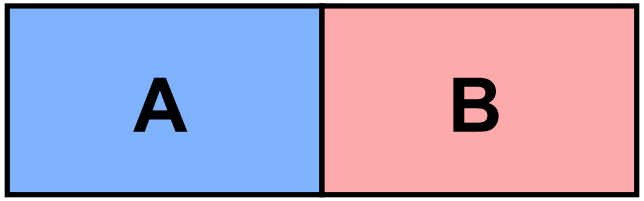
$$|\psi\rangle = \sum_{ab} c_{ab}|a\rangle|b\rangle, \quad c_{ab} \Rightarrow d_a \times d_b$$

Change basis for A and B using Schmidt (singular value) decomposition of state

$$|\psi\rangle = \sum_{i} s_i|i\rangle_A|i\rangle_B, \quad s_i \Rightarrow \min(d_a, d_b)$$

Less redundancy and ranks importance of contributions

A bipartite system



$$|\psi\rangle = \sum_i s_i |i\rangle_A |i\rangle_B$$

Distribution of singular values gives a measure of entanglement

$$\sum_i s_i^2 = 1, \quad S_E = -\sum_i s_i^2 \log s_i^2$$

Zero for product state, maximum log(N) for N states

Matrix Product States

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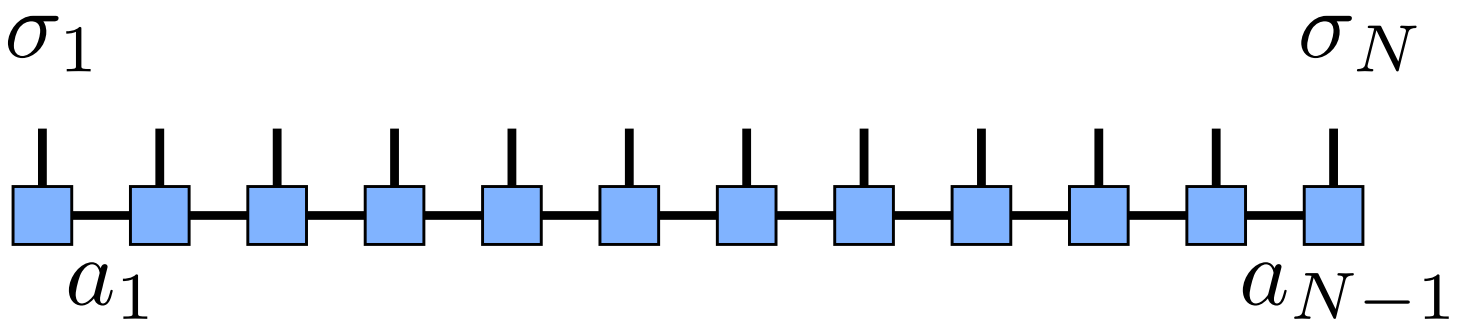
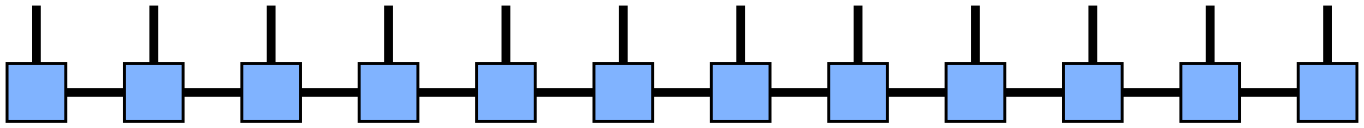
Carry out iterative SVD on a 1D system with N sites…..

$$|\psi\rangle = \sum_{\sigma} A^{\sigma_1}_{1 a_1} A^{\sigma_2}_{a_1 a_2} \cdots A^{\sigma_N}_{a_{N-1},1} |\sigma_1 \cdots \sigma_N\rangle$$



Matrix Product States

$$|\psi\rangle = \sum_{\sigma} A^{\sigma_1}_{1 a_1} A^{\sigma_2}_{a_1 a_2} \cdots A^{\sigma_N}_{a_{N-1},1} |\sigma_1 \cdots \sigma_N\rangle$$



- Restrict to largest D singular values at each bond
- Max independent coefficients for matrix dimension D?

$$(d_\sigma - 1) N D^2 \quad \text{vs.} \quad d_\sigma^N$$

- Max entanglement entropy at a bond is given by flat distribution of singular values

$$\log D$$

Density matrix renormalisation group

VOLUME 69, NUMBER 19 PHYSICAL REVIEW LETTERS 9 NOVEMBER 1992

Density Matrix Formulation for Quantum Renormalization Groups

Steven R. White
Department of Physics, University of California, Irvine, California 92717
(Received 22 May 1992)

PHYSICAL REVIEW B VOLUME 48, NUMBER 14 1 OCTOBER 1993-II

Density-matrix algorithms for quantum renormalization groups

Steven R. White
Department of Physics, University of California, Irvine, California 92717
(Received 13 January 1993)

- Iterative construction and optimisation of MPS with energy minimisation for 1D problems
- High accuracy, controlled by bond dimension D, (even near critical points, e.g. Pollmann et al. PRL 2009)

Density matrix renormalisation group

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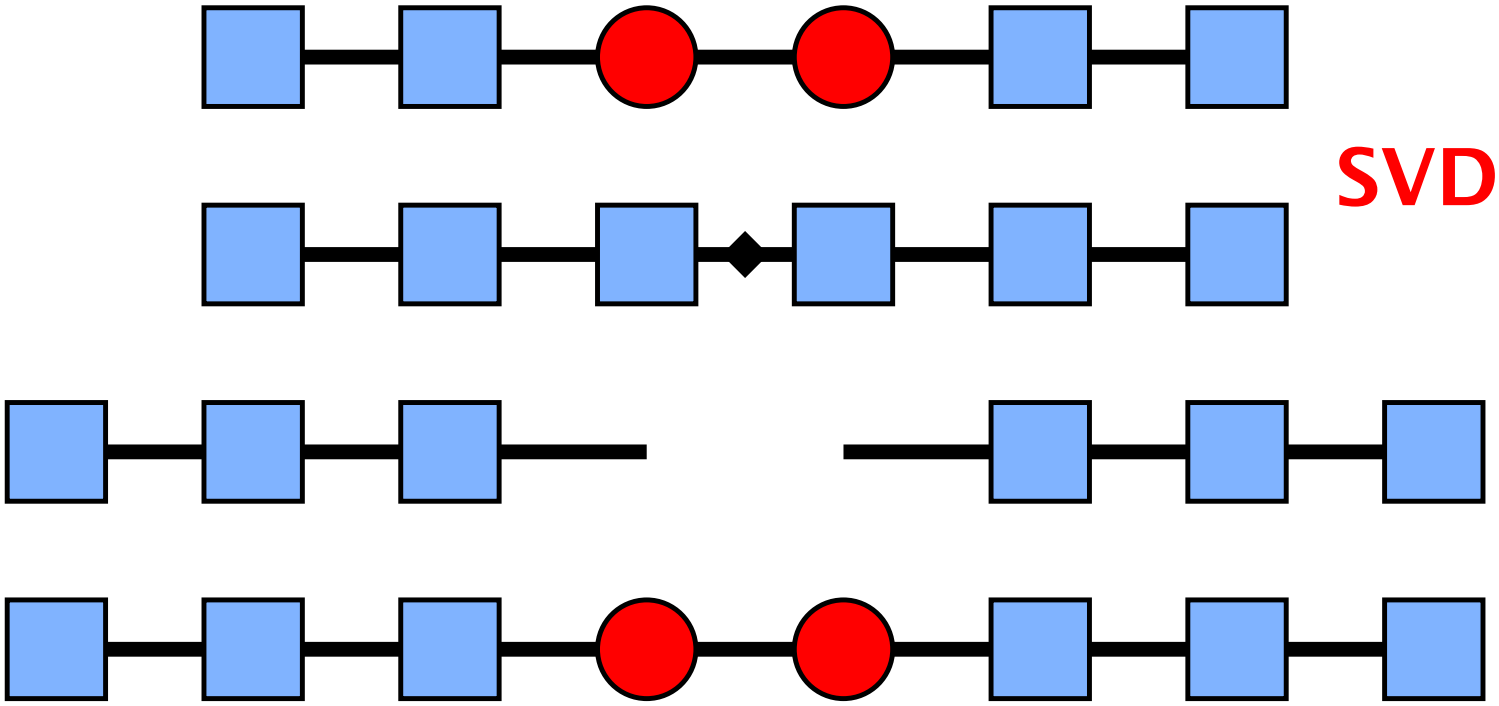
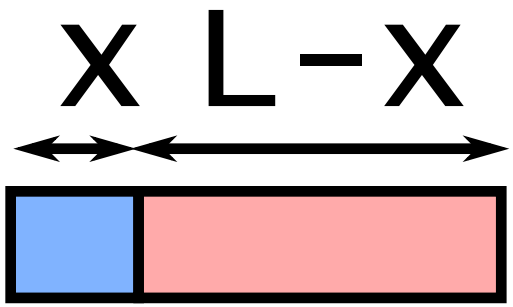


Why does it work?

Success of MPS methods in 1D relies on, at worst, log growth of entanglement

Bipartite 1D system



$$S_A = \frac{c}{6} \log L_{eff} + \cdots$$

$$L_{eff} = \begin{cases} \frac{L}{\pi} \sin \frac{\pi x}{L} & \text{critical} \\ \xi / a & \text{gapped} \end{cases}$$

Holzhey, Larsen & Wilczek 1994, Calabrese & Cardy 2004

How do we extend MPS to higher dimensions?

How does entanglement entropy behave for D>1?

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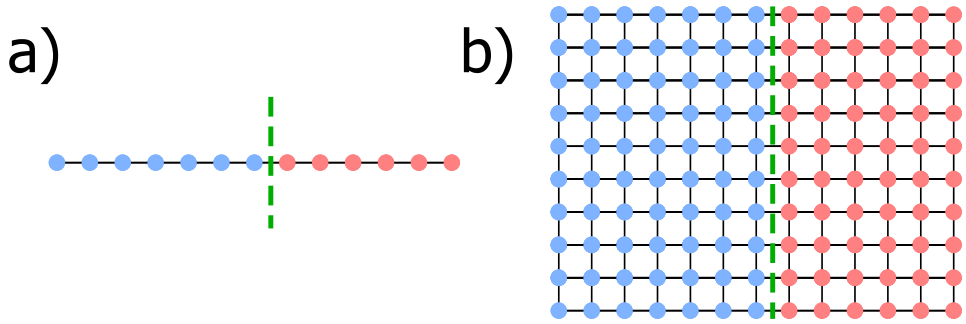
Generally an 'area law' is expected

$$S_E \sim \mathcal{A}/a^{D-1}$$



An exception: massless free fermions in D dimensional cube (Gioev and Klich 2006)

$$S_E \sim L^{D-1} \log L$$

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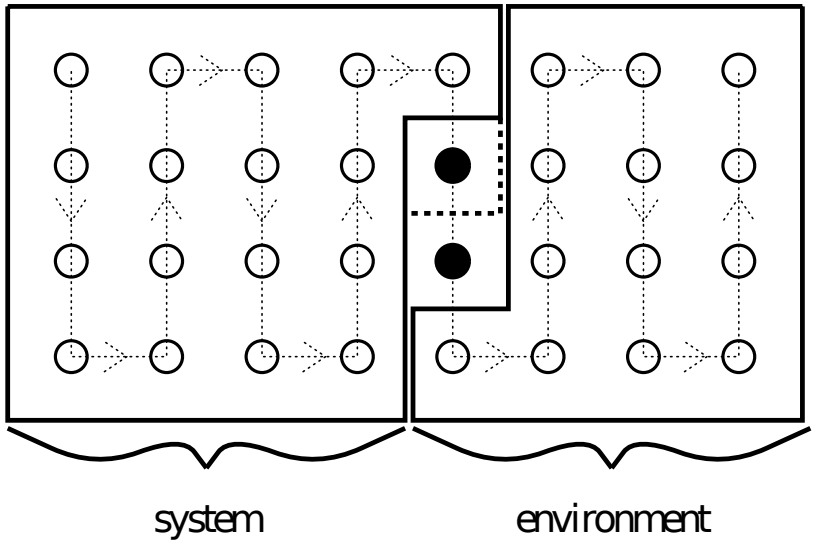
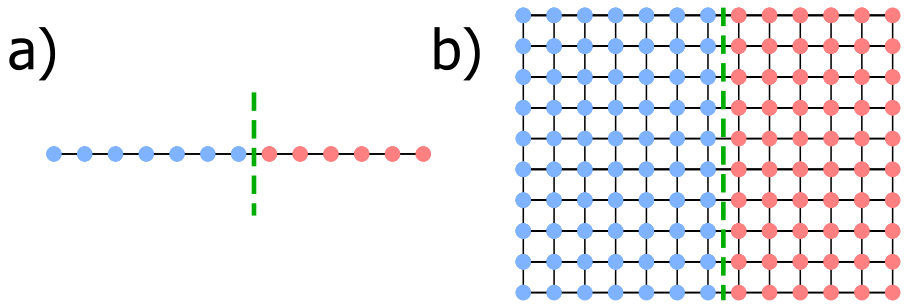
MPS methods less efficient in 2D than in 1D due to area law growth of entanglement





Effective long range interactions

General problem for tensor methods

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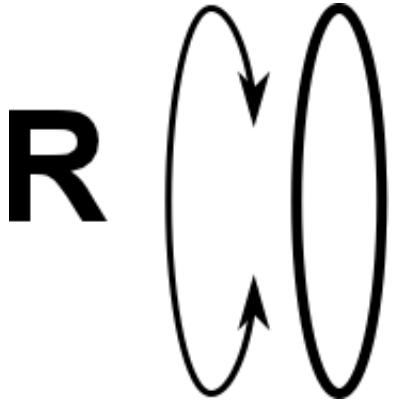
Truncated Spectrum Approach DMRG

Bending the area law

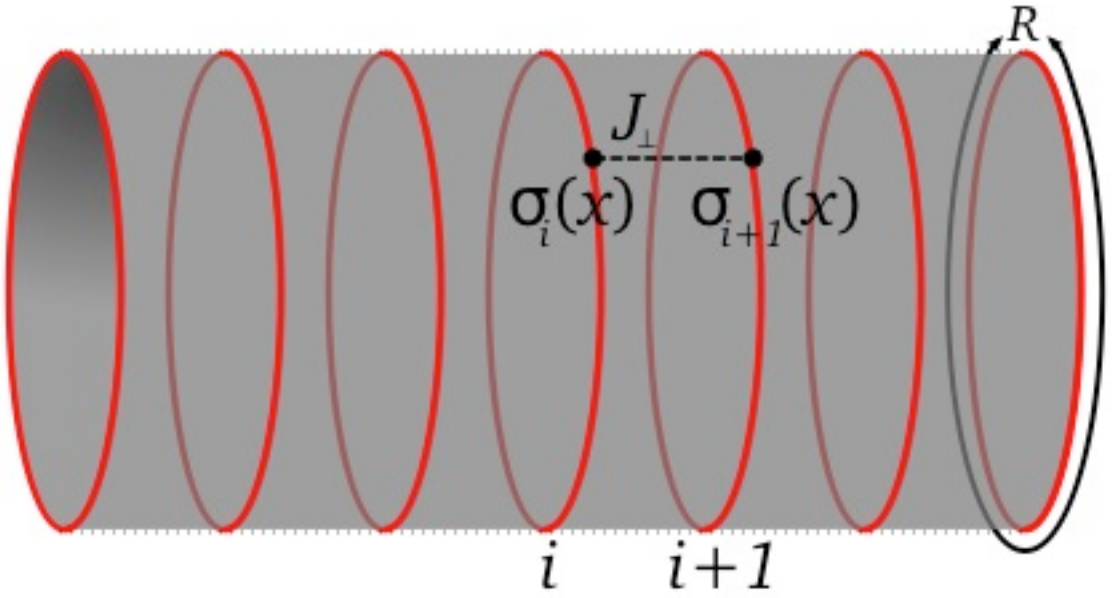
Chain, length R, periodic b.c.s, with continuum Hamiltonian that is either conformal or integrable



- Continuum limit: finite size corrections exponential, keep R small $\sim e^{-\Delta R}$
- Integrable or conformal: exact spectrum and matrix elements known
- Use chains as sites in normal 1D DMRG?

$$H = \sum_i H_i^{1D} + J_\perp \sum_{\langle ij \rangle} \mathcal{O}_i \mathcal{O}_j$$

Bending the area law



$$H = \sum_i H_i^{1D} + J_\perp \sum_{\langle ij \rangle} \mathcal{O}_i \mathcal{O}_j$$

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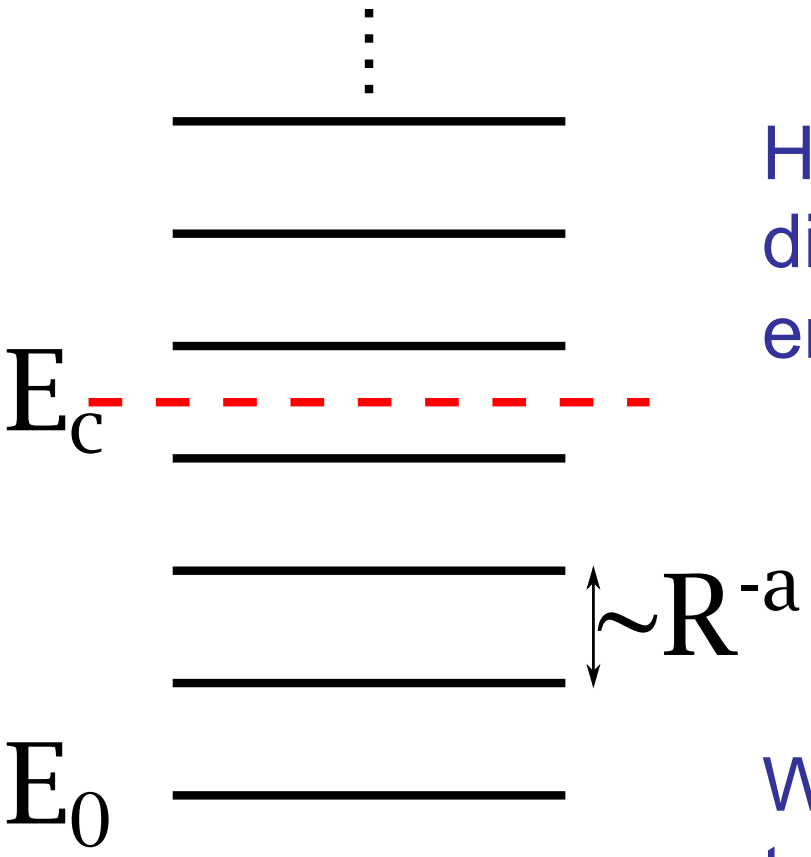
The spectrum of each chain is infinite…



However for finite R it is discrete, so we can order by energy and truncate

When can we expect/hope to get away with this?

Inspiration from TCSA

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Truncated Conformal Space Approach

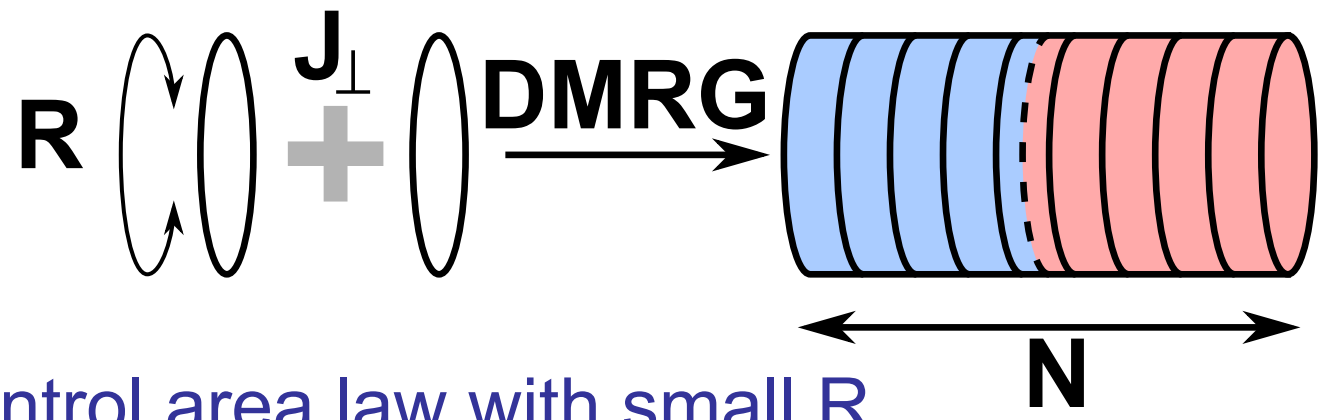
V. P. Yurov and Al. B. Zamolodchikov Int. J. Mod. Phys. A 6, 4557 (1991)

$$H = H_{CFT} + \lambda_0 \int d^2x\, \phi_0(x)$$

Relevant perturbing operator, most important effect is mixing of low energy states

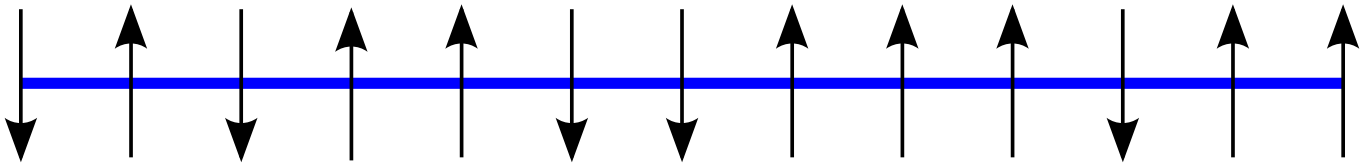
Critical Ising chain in magnetic field well described (error ~ 1%) by keeping only 39 states

Truncated spectrum approach to DMRG



- Control area law with small R
- Requires re-truncation in energy after each DMRG step to keep consistent cutoff
- In general don't need to keep too many reduced density matrix eigenvalues, ~30 in gapped phase to 100's nearer criticality for 10^{-5} truncation error

Example: Quantum Ising chain



- Continuum limit of lattice Ising chain in transverse field
- Can study negative mass, Δ, (disordered) or positive Δ (ordered) chains

$$H^{1D}_{\text{lattice}} = -J \sum_m \left(\sigma^z_m \sigma^z_{m+1} + (1+g)\sigma^x_m \right)$$

$$H^{1D} = \int dx \left[\frac{v}{2} \left(\bar{\Psi} \frac{\partial \bar{\Psi}}{\partial x} - \Psi \frac{\partial \Psi}{\partial x} \right) + \Delta \bar{\Psi} \Psi \right]$$

$$\Delta = -2gJ, v = 2Ja$$

Example: Quantum Ising chain

- Chain eigenstates organised into two ‘sectors’ Ramond and Neveu-Schwarz with integer and half-integer momenta

$$\text{NS: } |p_1, p_2, \cdots, p_{N_f}\rangle = \alpha^\dagger_{p_1}\alpha^\dagger_{p_2}\cdots\alpha^\dagger_{p_{N_f}}|0\rangle_{NS}, \quad p_i \in (\mathbb{Z}+1/2)\frac{2\pi}{R}$$

$$\text{R: } |k_1, k_2, \cdots, k_{N_f}\rangle = \alpha^\dagger_{k_1}\alpha^\dagger_{k_2}\cdots\alpha^\dagger_{k_{N_f}}|0\rangle_{R}, \quad k_i \in \mathbb{Z}\frac{2\pi}{R}$$

- Permissible states depend on sign of mass

$\Delta > 0$, NS states when N_f even, R states when N_f even.

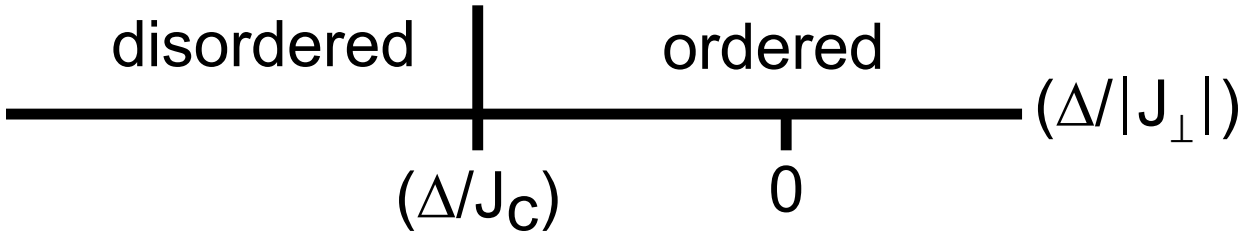
$\Delta < 0$, NS states when N_f even, R states when N_f odd.

- Fermion energies

$$E_{k_i} = \sqrt{\Delta^2 + k_i^2}$$

2D Quantum Ising model (2+1)

- Should be an order/disorder transition (disordered chains will order with enough coupling)



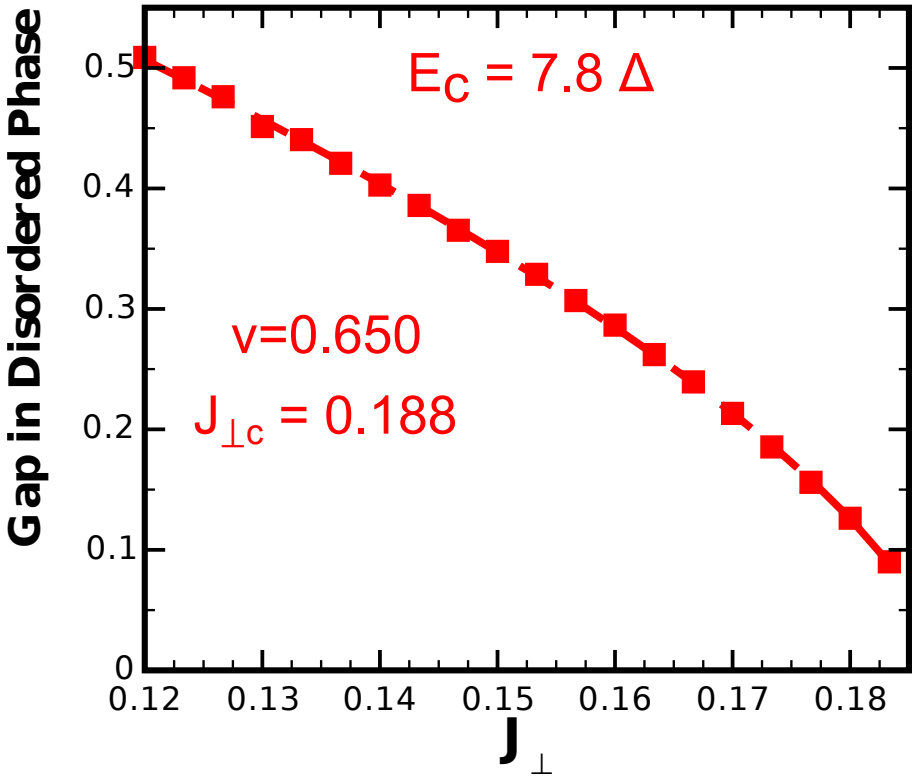
- Start with disordered (negative mass) chains and sweep chain coupling, study the energy gap
- Test that the correct many-body behaviour is captured

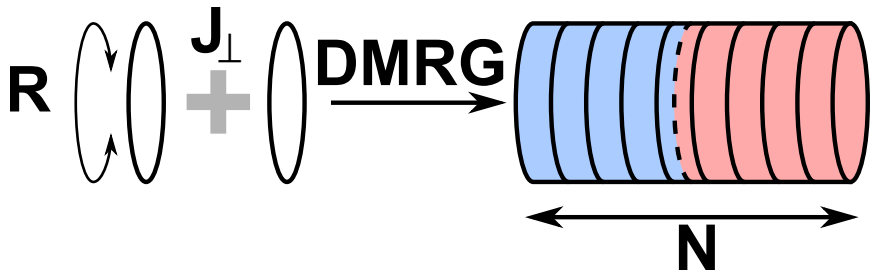
2D Quantum Ising model (2+1)

$$\Delta_{2DQI} \sim |J_c - J_\perp|^\nu$$

$$N = 60,\ R\Delta = 10,\ E_c = 7.8\Delta$$

R. M. Konik and Y. Adamov, PRL 102, 097203 (2009)



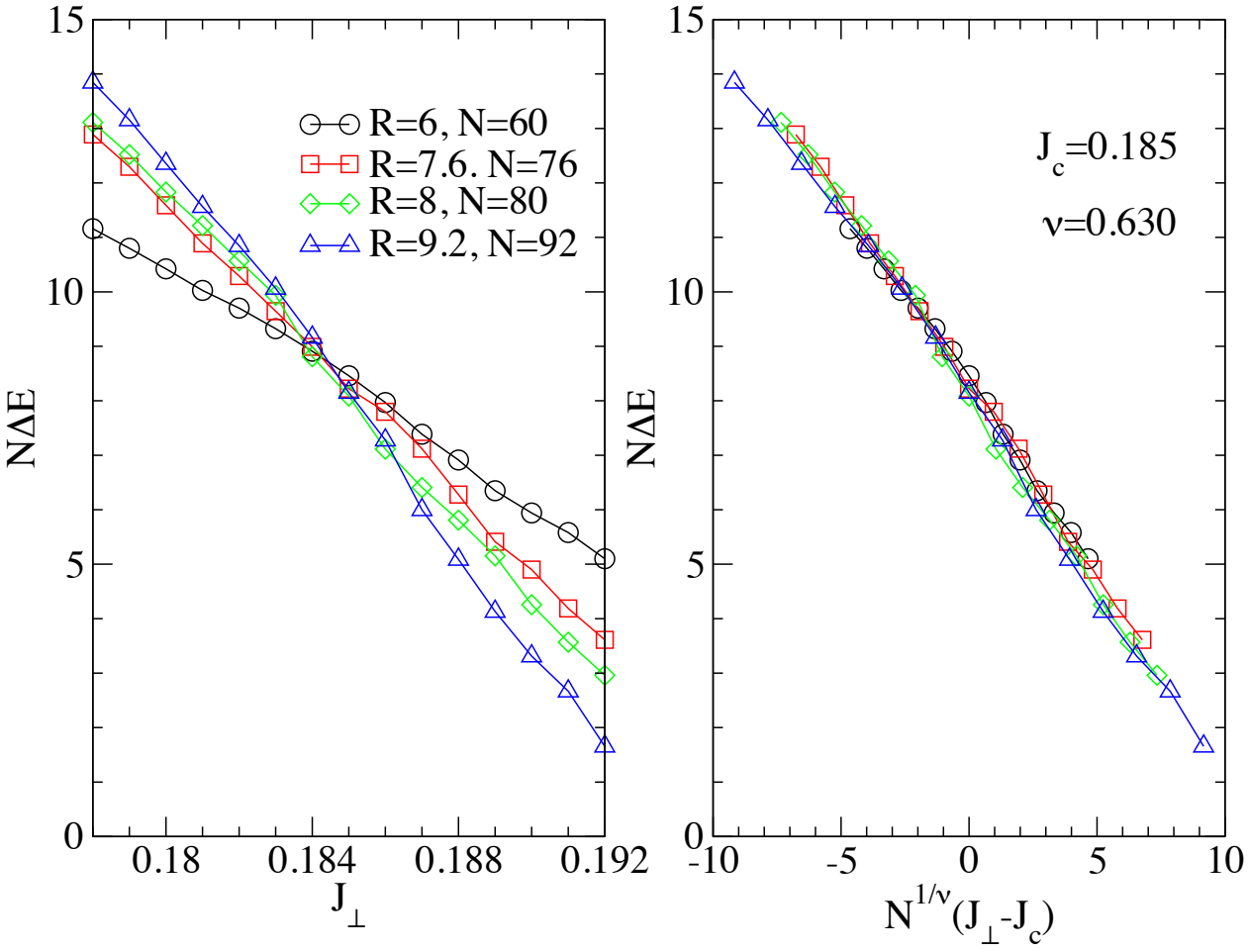


In 3D classical Ising universality class,

$$\nu = 0.630$$

2D Quantum Ising model (2+1)

Finite size scaling $\nu = 0.630$



Other things to look at

- Entanglement (James, Konik PRB 87, 2013)
- Generalise to other chain Hamiltonians
  - Coupled XXZ
  - Coupled Hubbard chains
- Infinite DMRG for thermodynamic limit
- Take advantage of other MPS/MPO algorithms
  - Time evolution and iTEBD
  - Finite temperature

Time evolution and out-of-equilibrium

Quantum quenches

- Take an eigenstate of some ‘initial’ Hamiltonian and time evolve it with respect to a new Hamiltonian

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

- How do we approach stationary state and thermalise?
- What is the return probability (Loschmidt echo)

$$\mathcal{L}(t) = |\langle\psi(0)|e^{iH_0 t}e^{-iHt}|\psi(0)\rangle|^2$$

- In 1D, can use MPS based time evolution and analytic methods

Quantum quenches in cold atoms

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

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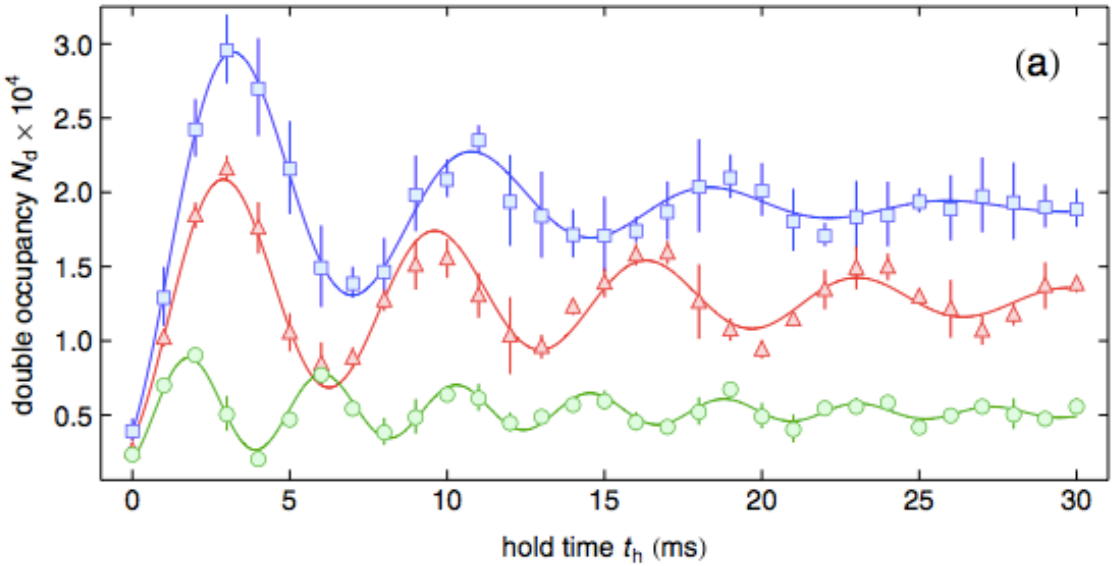
Quantum Quench in an Atomic One-Dimensional Ising Chain

F. Meinert,[1] M. J. Mark,[1] E. Kirilov,[1] K. Lauber,[1] P. Weinmann,[1] A. J. Daley,[2] and H.-C. Nägerl[1]
[1]*Institut für Experimentalphysik und Zentrum für Quantenphysik, Universität Innsbruck, 6020 Innsbruck, Austria*
[2]*Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260, USA*
(Received 9 April 2013; published 31 July 2013)

We study nonequilibrium dynamics for an ensemble of tilted one-dimensional atomic Bose-Hubbard chains after a sudden quench to the vicinity of the transition point of the Ising paramagnetic to antiferromagnetic quantum phase transition. The quench results in coherent oscillations for the orientation of effective Ising spins, detected via oscillations in the number of doubly occupied lattice sites. We characterize the quench by varying the system parameters. We report significant modification of the tunneling rate induced by interactions and show clear evidence for collective effects in the oscillatory response.

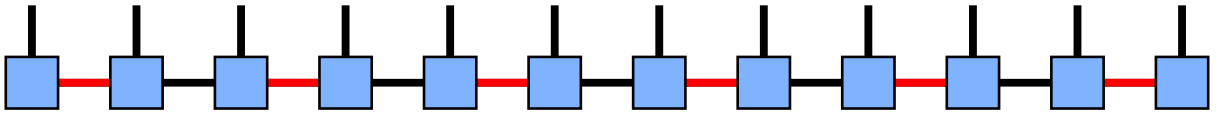


Time evolving block decimation

- Trotter decomposition of time steps into odd and even bond operations

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$
$$= \prod_n e^{-iHt_n}|\psi(0)\rangle, \quad \sum_n t_n = t$$

$$e^{-iHt_n} \approx e^{-iH_{odd}t_n} e^{-iH_{even}t_n}$$



$$e^{-iH_{odd}t_n} = e^{-iH_{1,2}t_n} e^{-iH_{3,4}t_n} \ldots e^{-iH_{N-1,N}t_n}$$

Time evolving block decimation

- Trotter decomposition of time steps into odd and even bond operations
- Compress MPS after time step
- Time step error can be controlled with smaller steps or higher order decompositions
- Error due to truncation is more problematic

Time evolving block decimation

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- Time step error can be controlled with smaller steps or higher order decompositions
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Entanglement grows (at worst) linearly with time (Lieb Robinson bounds), P. Calabrese and J. Cardy (2006)

MPS dimensions grow (at worst) exponentially

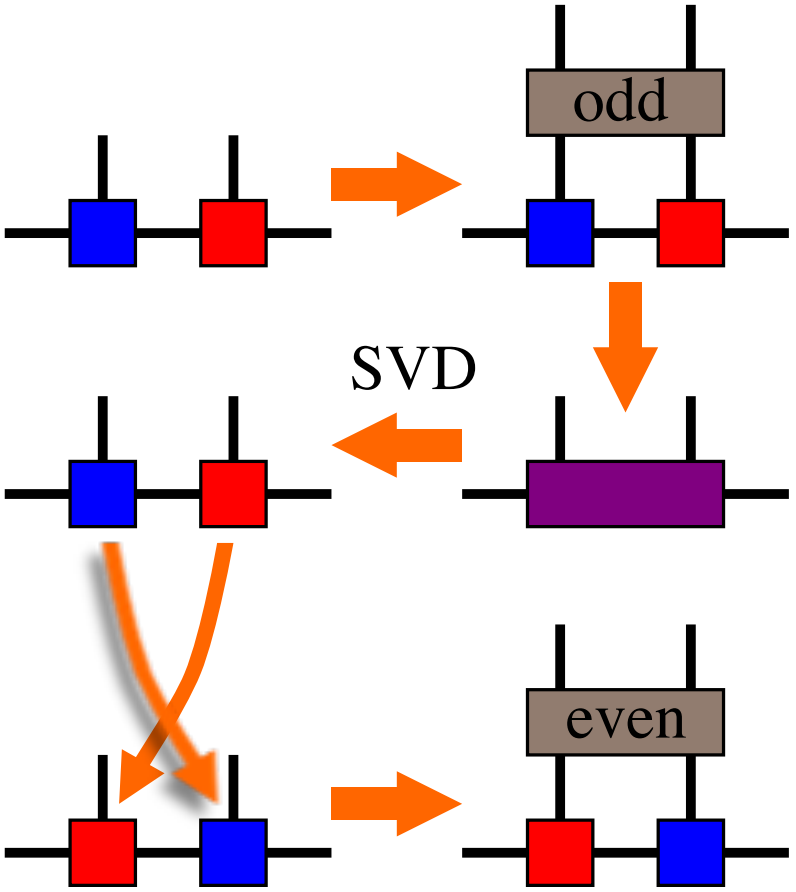
$$S_E(t) \leq S_E(0) + ct$$

$$D \lesssim \#^S \Rightarrow D \lesssim \#^t$$

Infinite MPS in 1D: iTEBD

Infinite time evolving block decimation

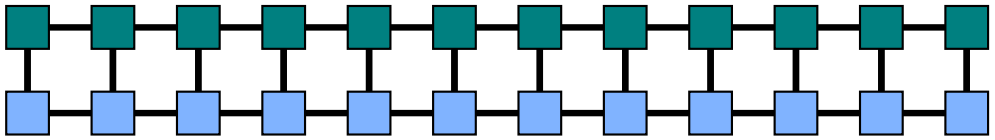
- Impose translational invariance
- Work with two site fragment
- Fast compared to finite chains (only 2 matrices instead of N)
- Better numerical stability than for large finite N



Infinite MPS in 1D: iTEBD

For a finite system we can do the full contraction over MPS indices

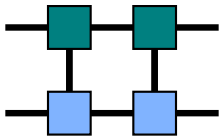
$$\langle\tilde{\psi}|\psi\rangle$$



For the infinite case we must work with the transfer operator

$$T_{(\tilde{a}_l a_l),(\tilde{a}_{l+2} a_{l+2})}$$

$$= \{\tilde{A}^{\sigma_{l+1}*}_{\tilde{a}_{l+2},\tilde{a}_{l+1}} \tilde{A}^{\sigma_l *}_{\tilde{a}_{l+1},\tilde{a}_l}\}\{A^{\sigma_l}_{a_l,a_{l+1}} A^{\sigma_{l+1}}_{a_{l+1},a_{l+2}}\}$$



Overlap/site is given by (square root of) the dominant eigenvalue

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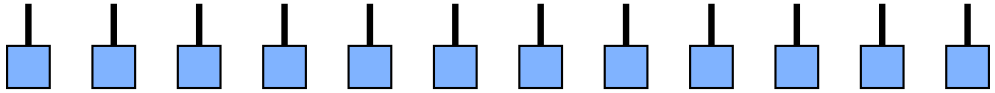
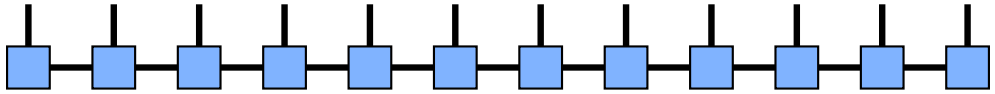
Quantum quenches for coupled chains

Global quench setup

- Start with product state (interchain coupling zero), for disordered (negative mass) chains
- Know ground state exactly, D=1, S_E=0, MPS is easy to form



- Turn on small interchain coupling



- Evolve in time

Perturbative treatment of a quench

- As a check, consider very shallow quenches that can be treated perturbatively (*in particular, do not cross critical coupling*)

$$H = H_0 + gH_1, \quad g \ll 1$$

- Need unitary perturbation theory to avoid terms that grow in time without bound

$$|\phi(t)\rangle = e^{-iHt}|\phi(0)\rangle = e^{-S}e^{-iH_{diag}t}e^{S}|\phi(0)\rangle$$

Moeckel and Kehrein, PRL 100, 175702 (2008)
Kollar, Wolf and Eckstein, PRB 84, 054304 (2011)

Perturbative treatment for coupled Ising chains

- Expectation of an operator on chain *i* at time *t*

$$\langle \mathcal{O}_i \rangle_t = \langle \mathcal{O}_i \rangle_0 + 8 J_\perp^2 R^2 \sum_{n_i, n_{i+1}} \left(\frac{\sin^2(t[E_i + E_{i+1}]/2)}{(E_i + E_{i+1})^2} \times \delta_{k_i, -k_{i+1}} |\langle 0|\sigma_i|n_i\rangle \langle 0|\sigma_{i+1}|n_{i+1}\rangle|^2 \langle n_i|\mathcal{O}_i|n_i\rangle \right)$$

- States of system are tensor products of states on individual chains

$$|\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_N\rangle$$

$$|\Phi(0)\rangle = |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$$

$$H_{1D,i}|\phi_i\rangle = E_{\phi_i}|\phi_i\rangle$$

Perturbative treatment for coupled Ising chains

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$$\langle \mathcal{O}_i \rangle_t = \langle \mathcal{O}_i \rangle_0 + 8 J_\perp^2 R^2 \sum_{n_i, n_{i+1}} \left(\frac{\sin^2(t[E_i + E_{i+1}]/2)}{(E_i + E_{i+1})^2} \right.$$

$$\left. \times \delta_{k_i, -k_{i+1}} |\langle 0|\sigma_i|n_i\rangle \langle 0|\sigma_{i+1}|n_{i+1}\rangle|^2 \langle n_i|\mathcal{O}_i|n_i\rangle \right)$$

Key features at 2nd order:

- No dependence on number of chains or boundary (except trivial factor of 2 at edges if O.B.C.)
- Sum of terms periodic in *t*, overlap with initial state is trivial to this order:

$$\mathcal{O} = |0\rangle\langle 0| \Rightarrow \langle \mathcal{O} \rangle_t = 1$$

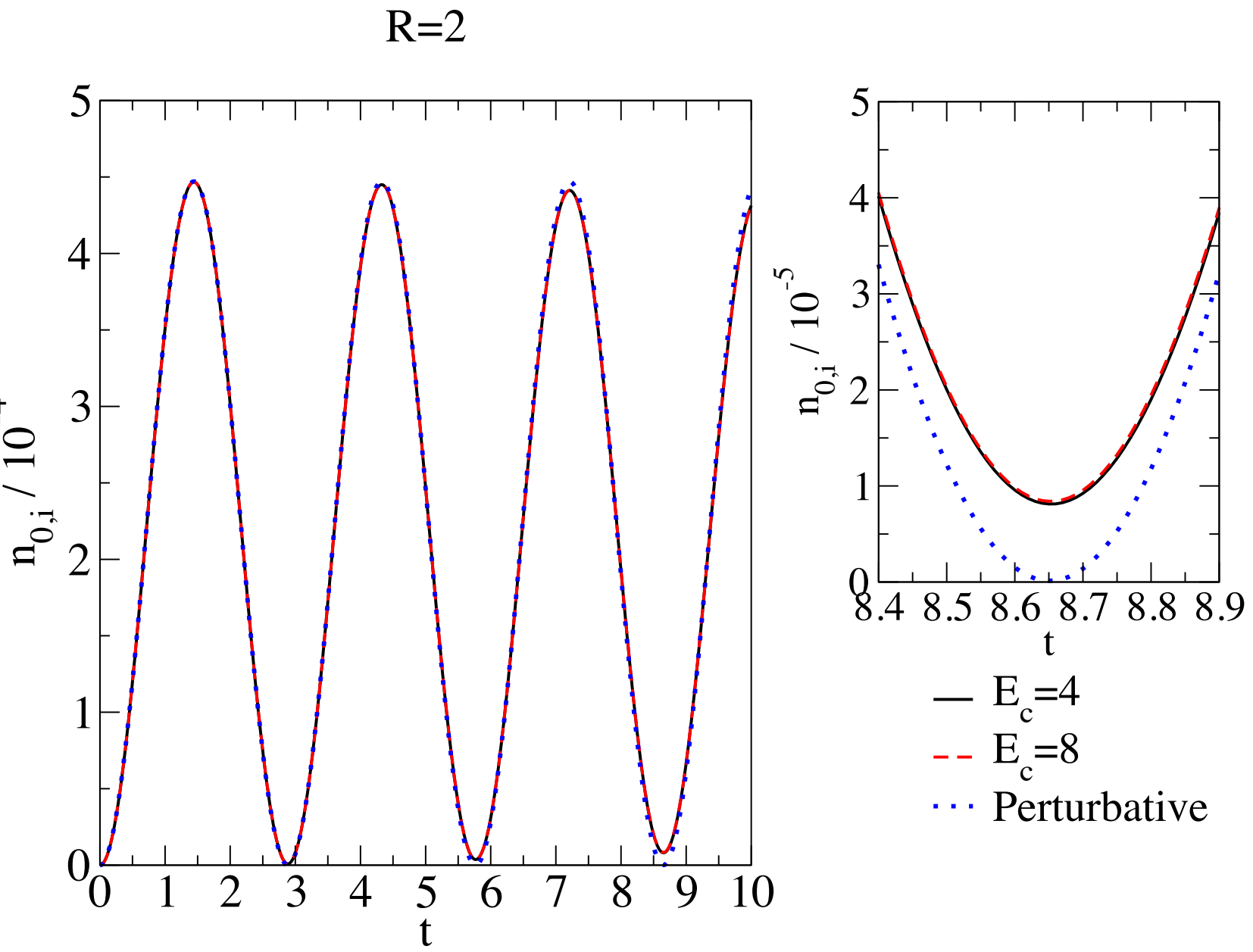
Chain momentum occupation number

$$n_{k,i}$$

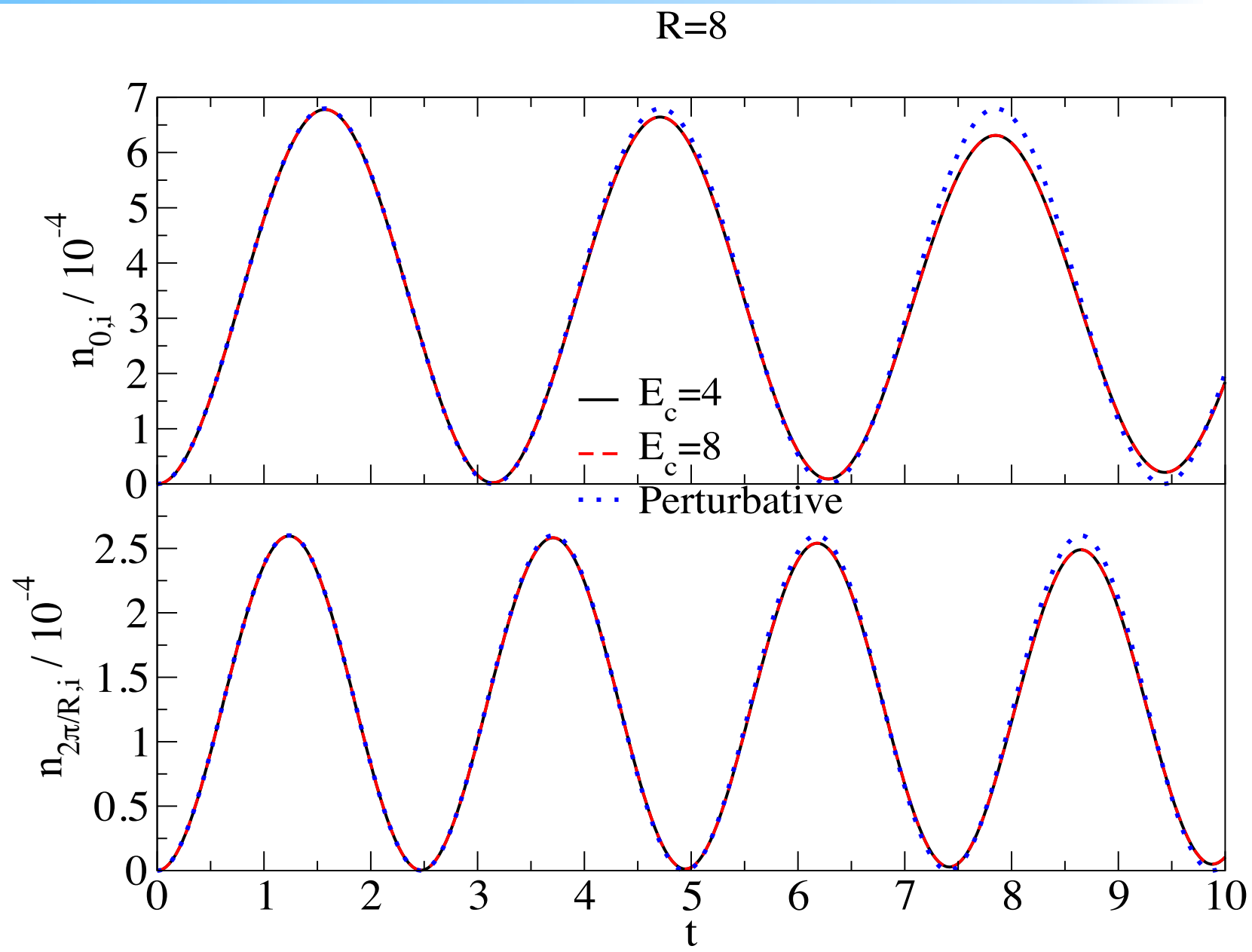
Consider number operator for excitations on chain i with chain momentum k

- Chain excitations are fermions, maximum value is 1 for each (k,i) pair
- Gives a measure of how far each chain is from its (uncoupled) groundstate

Small quench $J_\perp = 0.01 \ll J_c \sim 0.18$

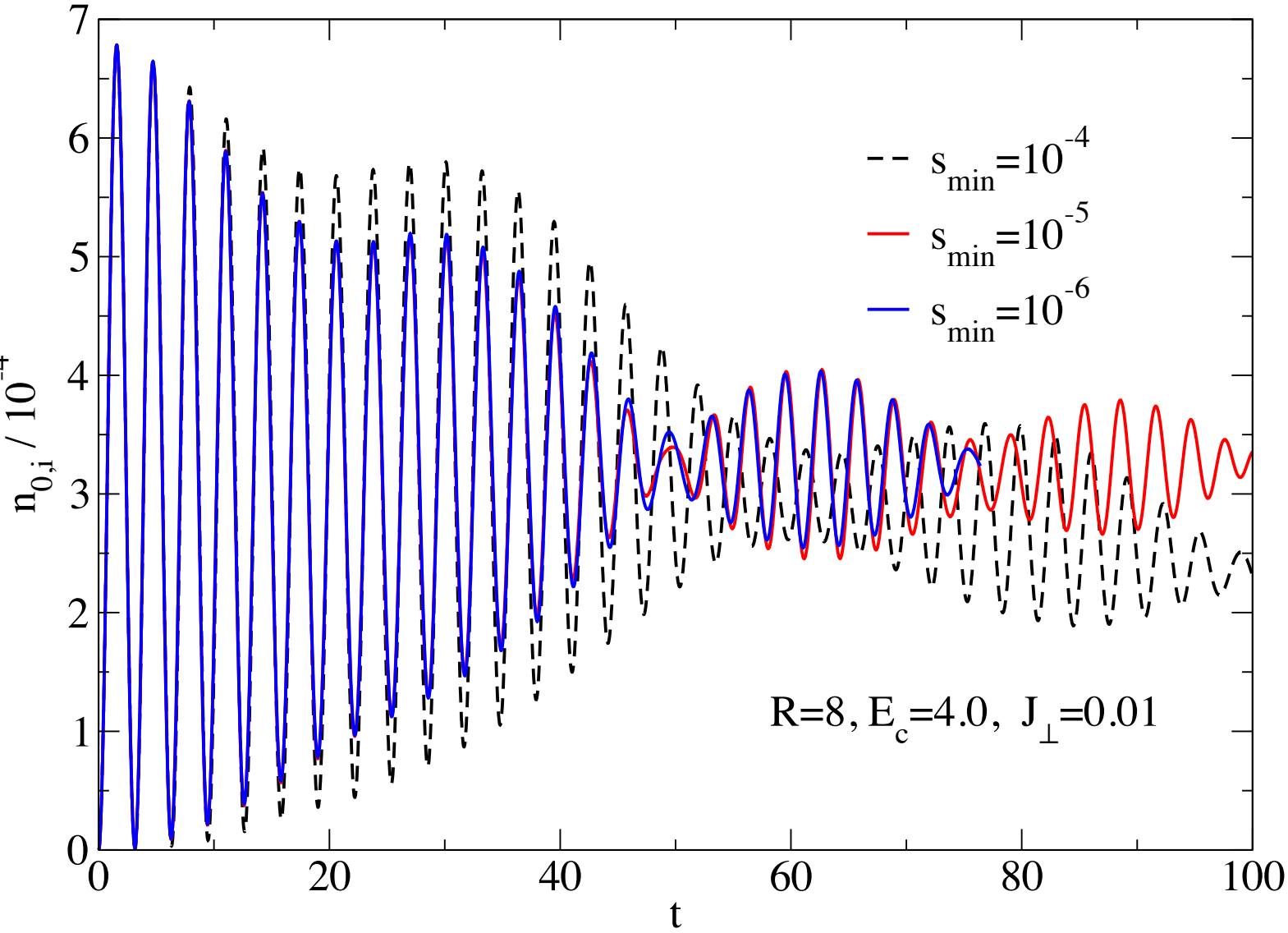


Small quench $J_\perp = 0.01 \ll J_c \sim 0.18$



Longer times

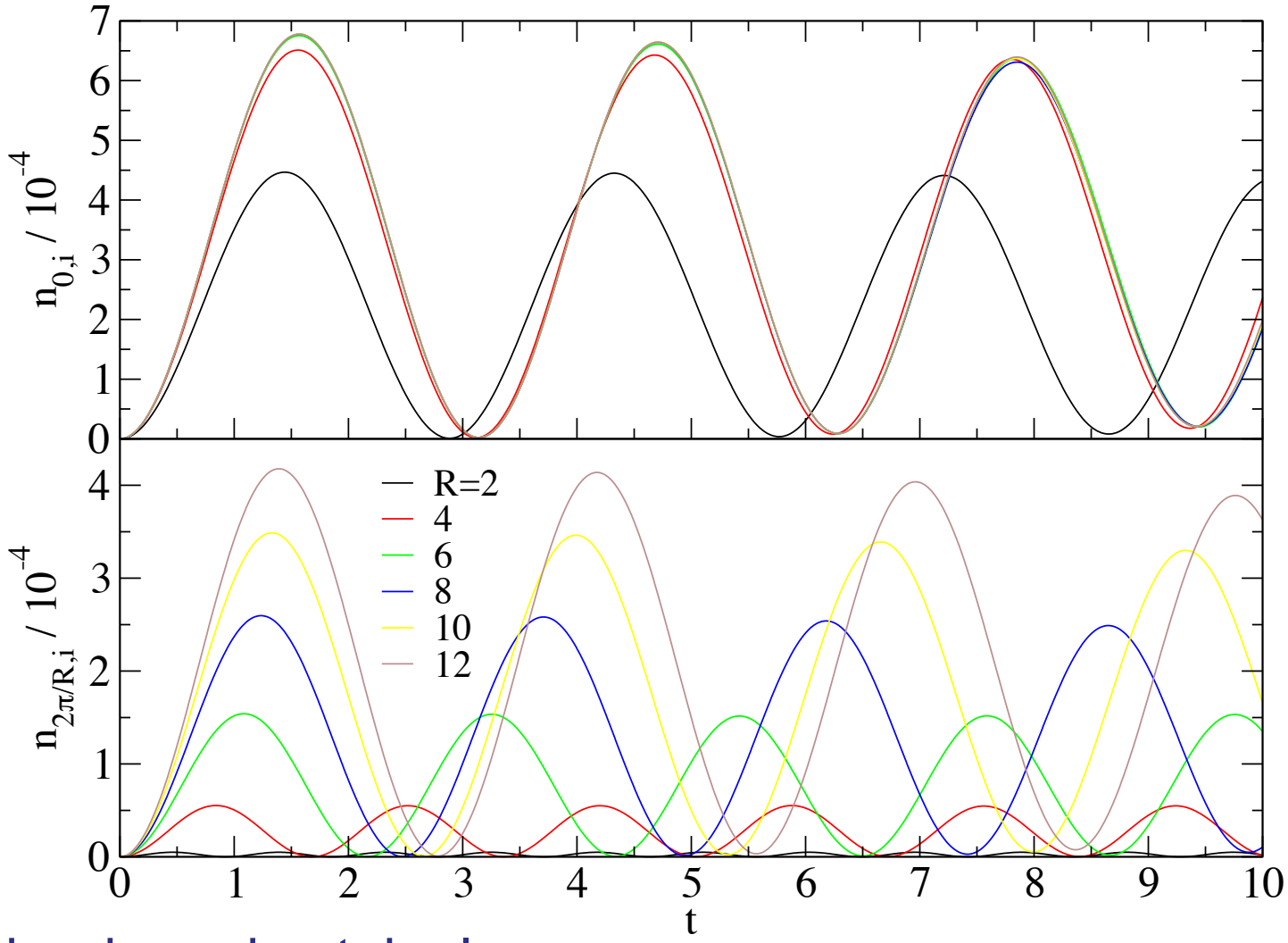
$$J_\perp = 0.01 \ll J_c \sim 0.18$$



Momentum modes versus R

LCN
LONDON CENTRE FOR NANOTECHNOLOGY

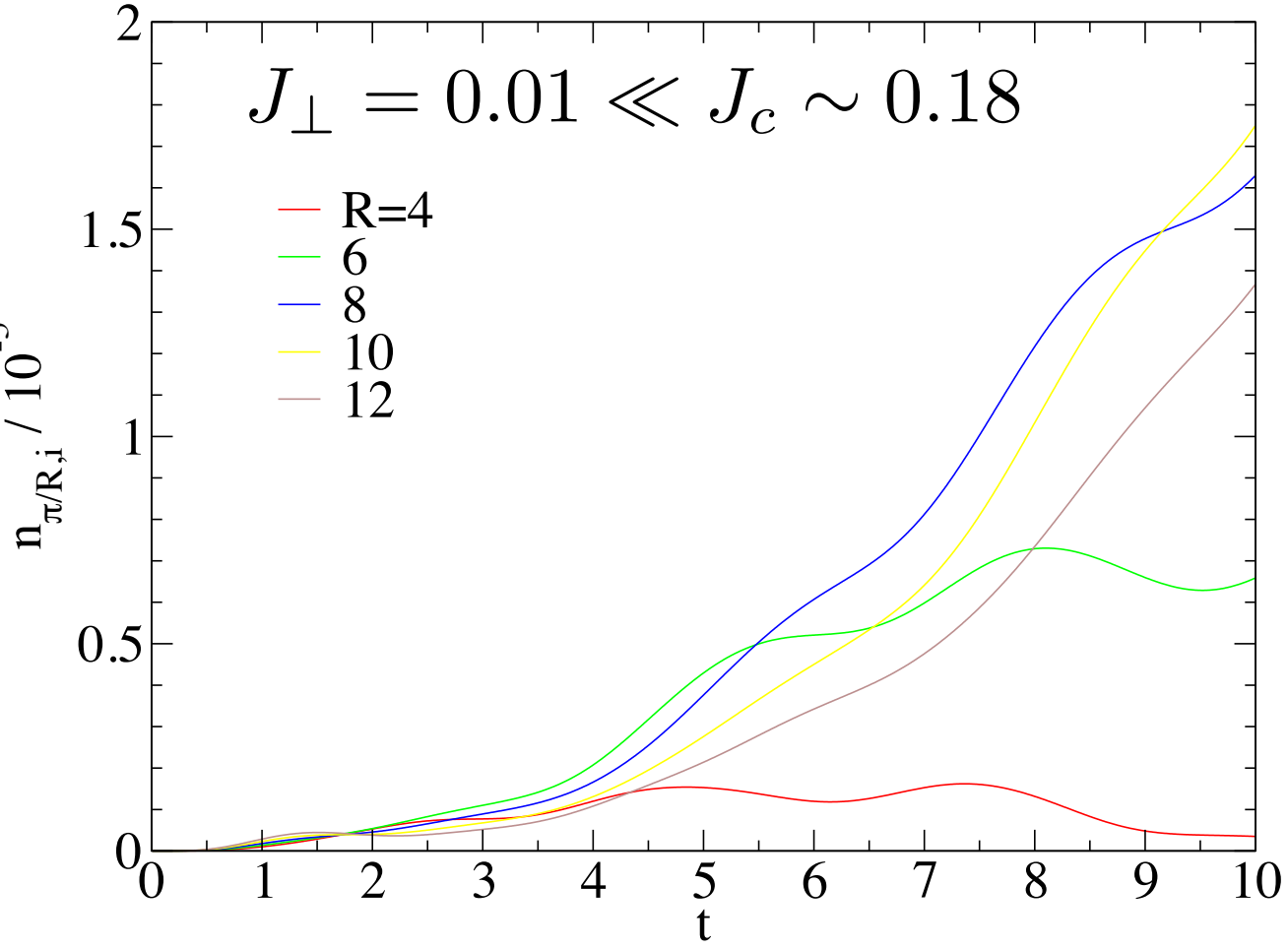
Chain k=0 mode saturates quickly with R



Higher k modes take longer…

Half integer chain momentum modes

$$k = \frac{2\pi n}{R}, \quad n \in \mathbb{Z} + 1/2$$ NS half integer modes



Not captured by the perturbative result

Dynamics of overlap

$$G(t) = \langle \Psi_0 | e^{-iHt} | \Psi_0 \rangle$$

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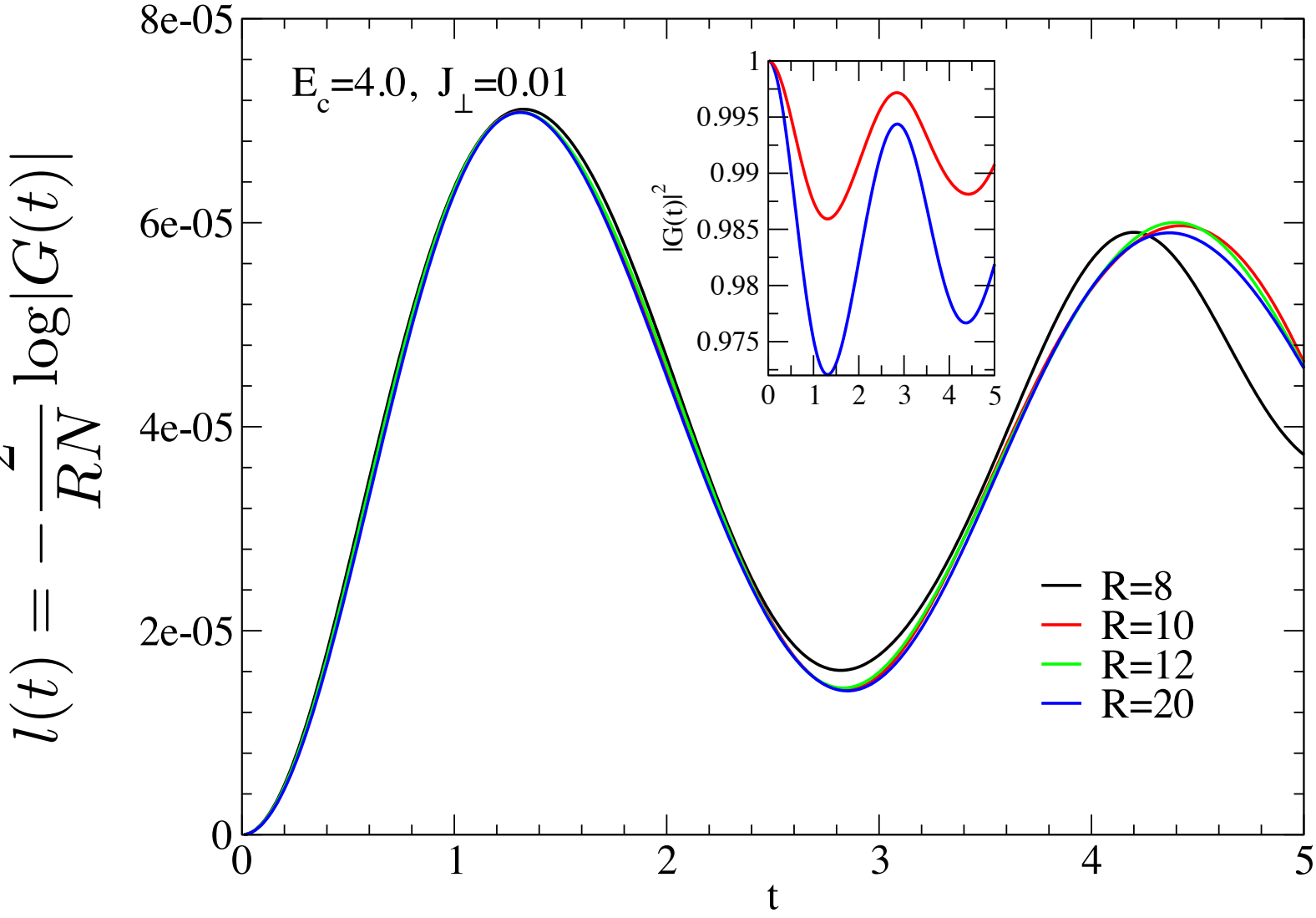
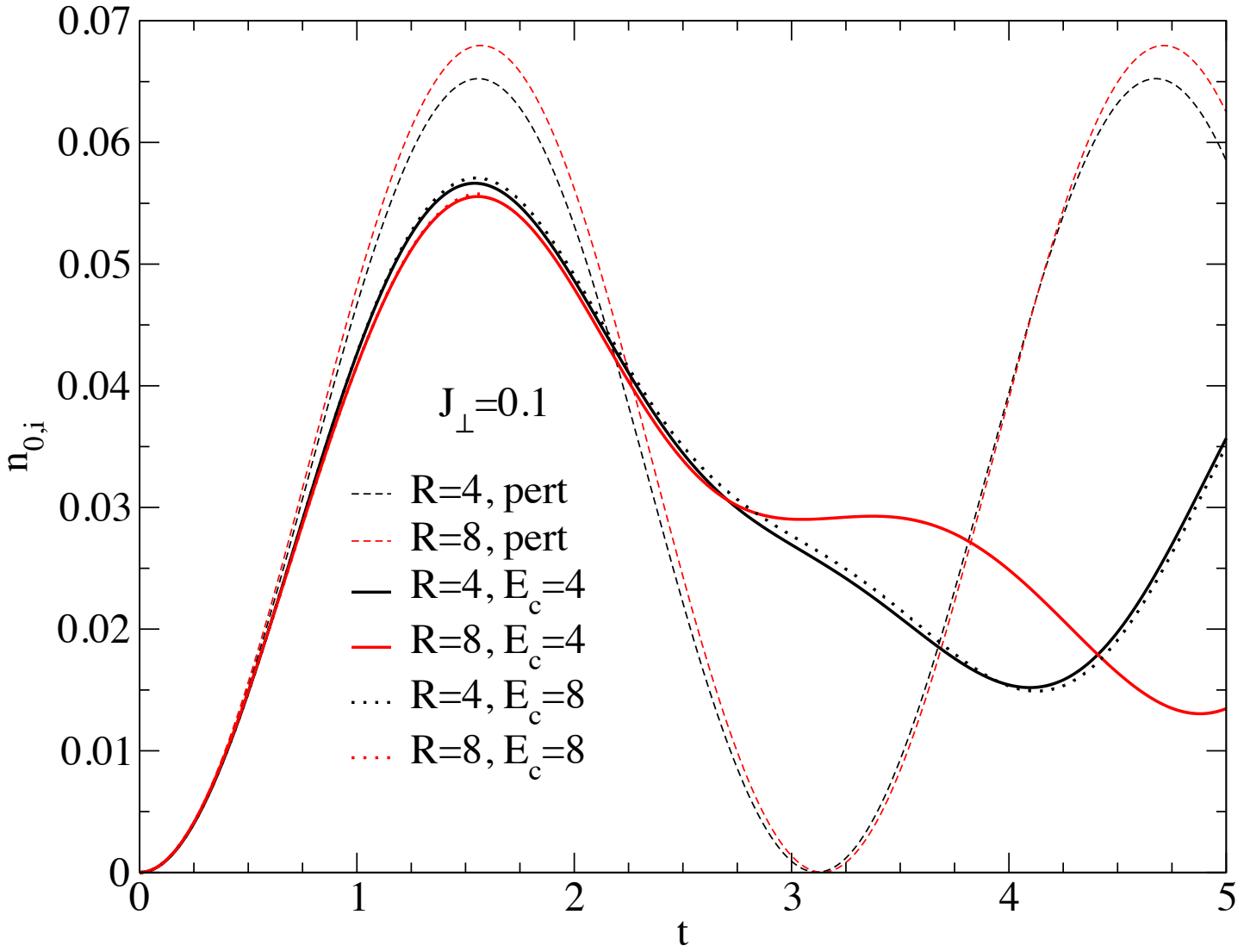
c.f. trivial overlap to 2nd order in perturbative calculation



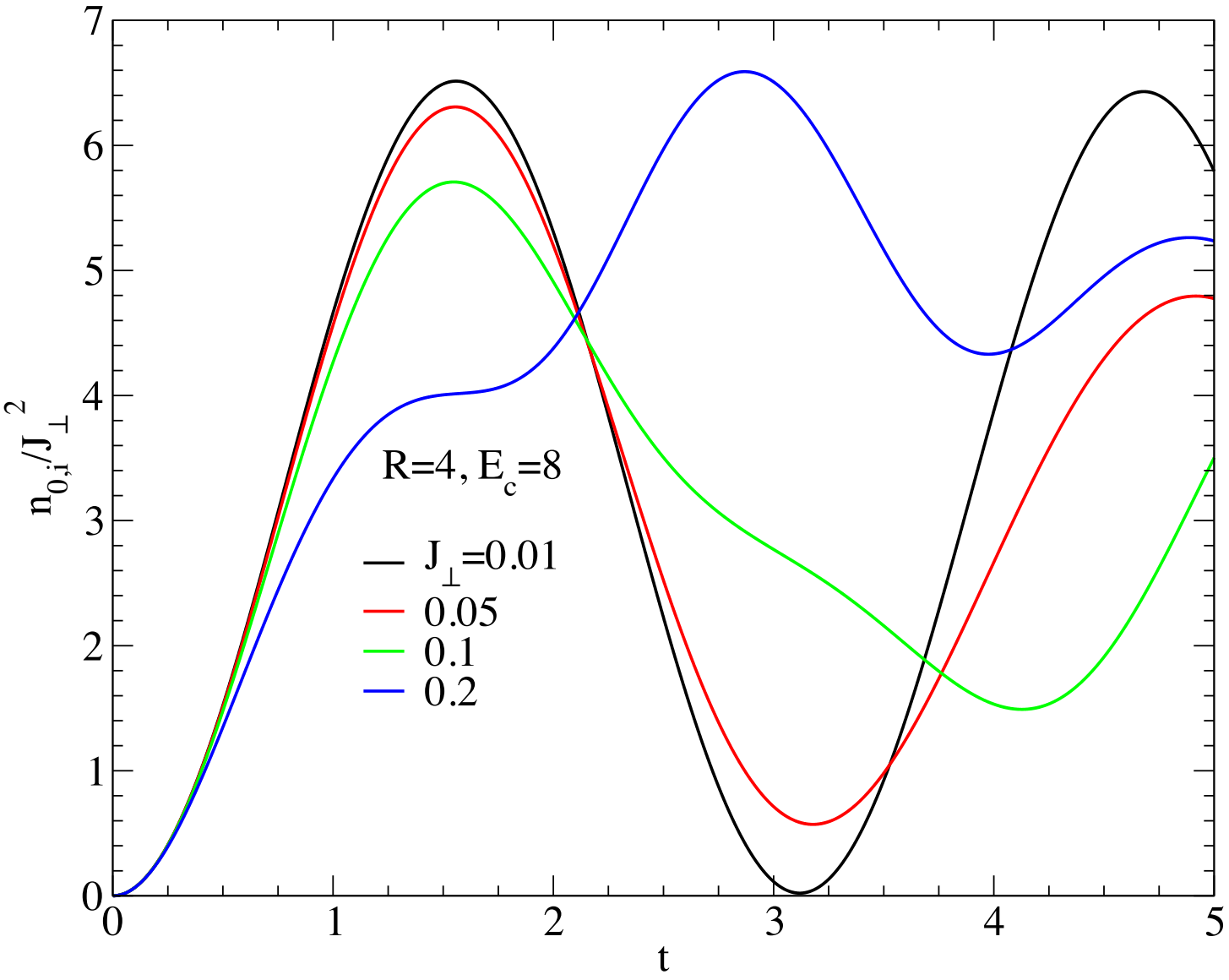
Deeper quenches for coupled chains

Moving away from the perturbative limit

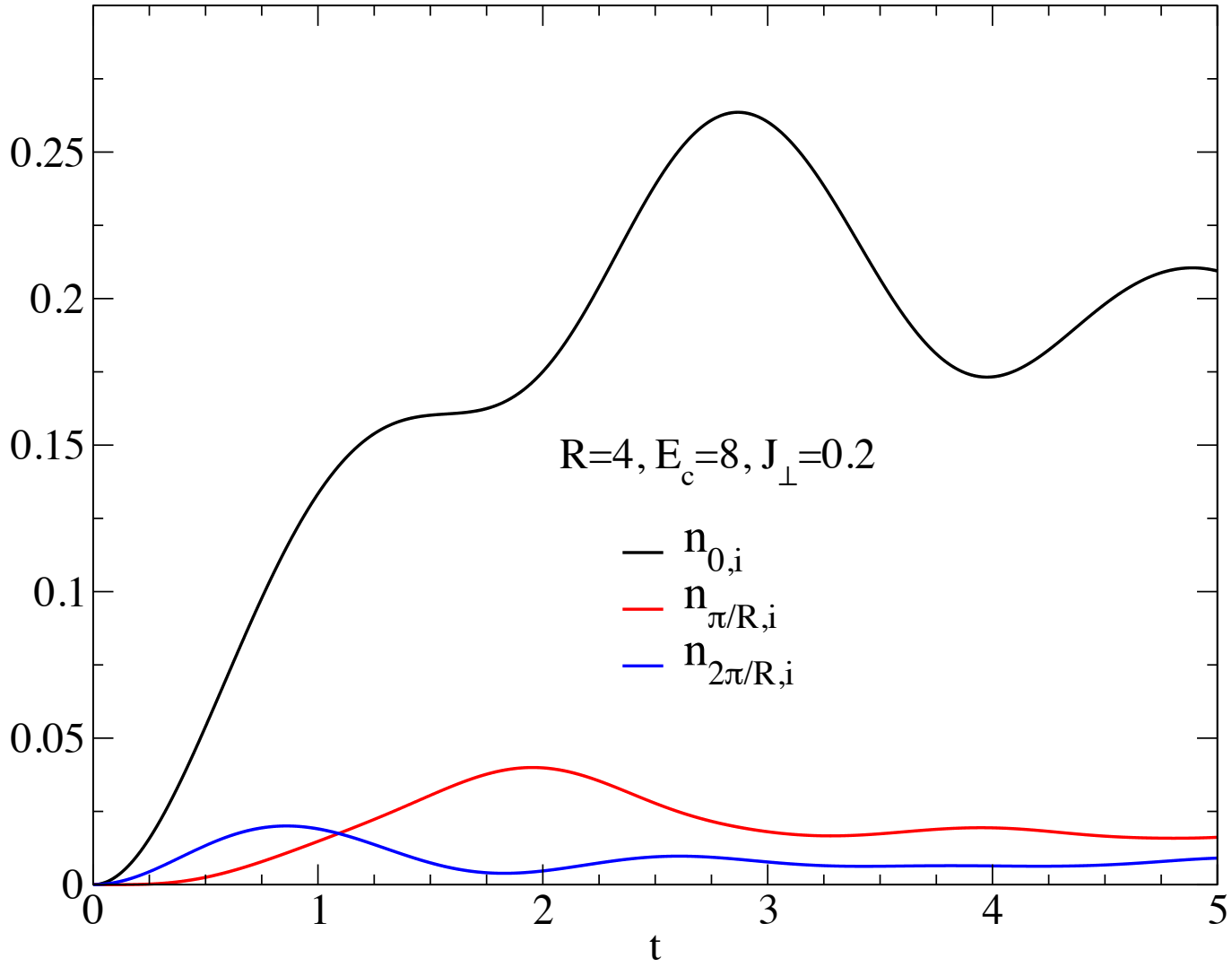


Departure from simple oscillatory behaviour

Moving away from the perturbative limit



Moving away from the perturbative limit



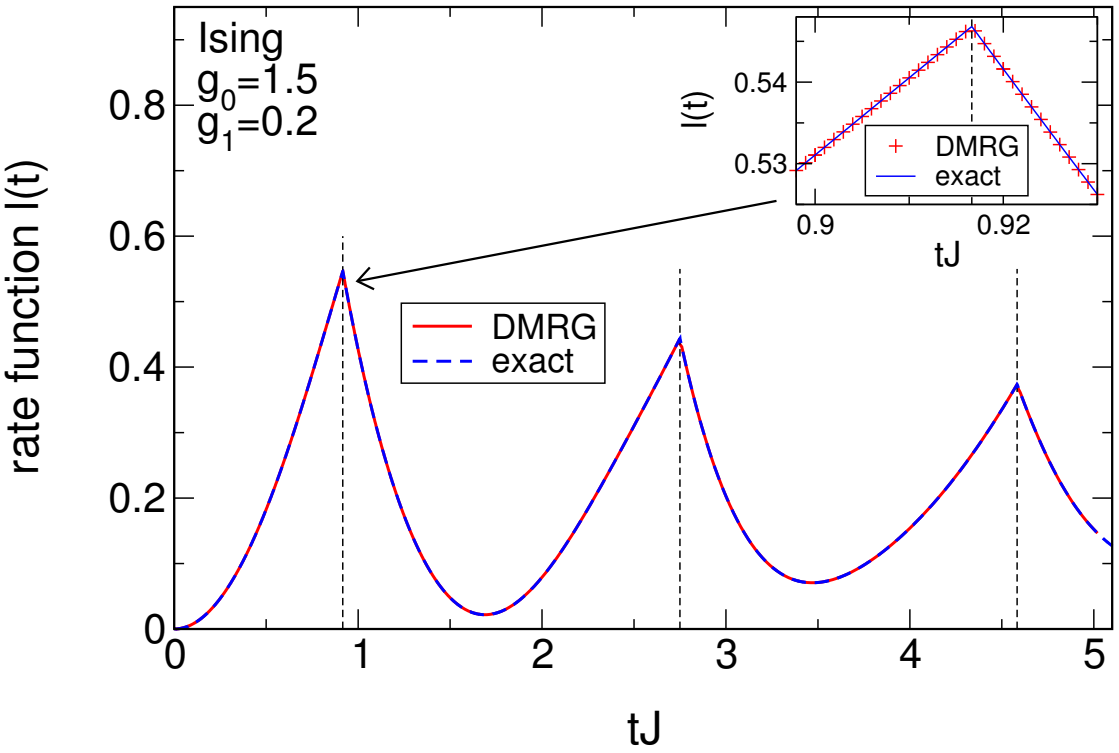
Half integer modes become ‘large’

Dynamical Phase Transitions?

Heyl, Polkovnikov and Kehrein, PRL 110, 135704 (2013)

For TFIM, nonanalytic 'rate' if quench is through critical point

$$l(t) = -\lim_{N\to\infty} \frac{1}{N} \log|G(t)|^2$$



Karrasch and Schuricht
PRB 87, 195104 (2013)

Dynamical Phase Transitions?

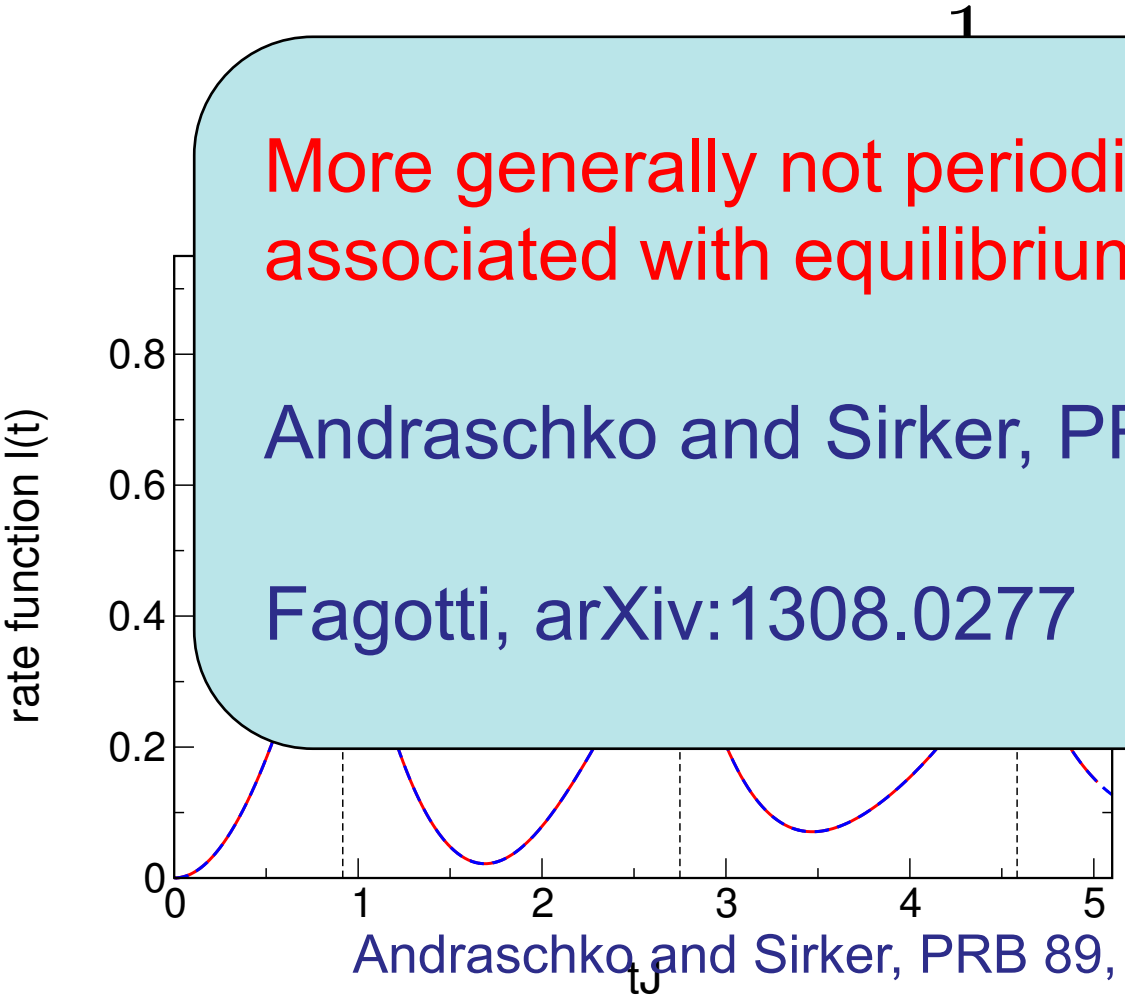
Heyl, Polkovnikov and Kehrein, PRL 110, 135704 (2013)

For TFIM, nonanalytic 'rate' if quench is through critical point

More generally not periodic, nor uniquely associated with equilibrium critical points

Andraschko and Sirker, PRB 89 (2014)

Fagotti, arXiv:1308.0277

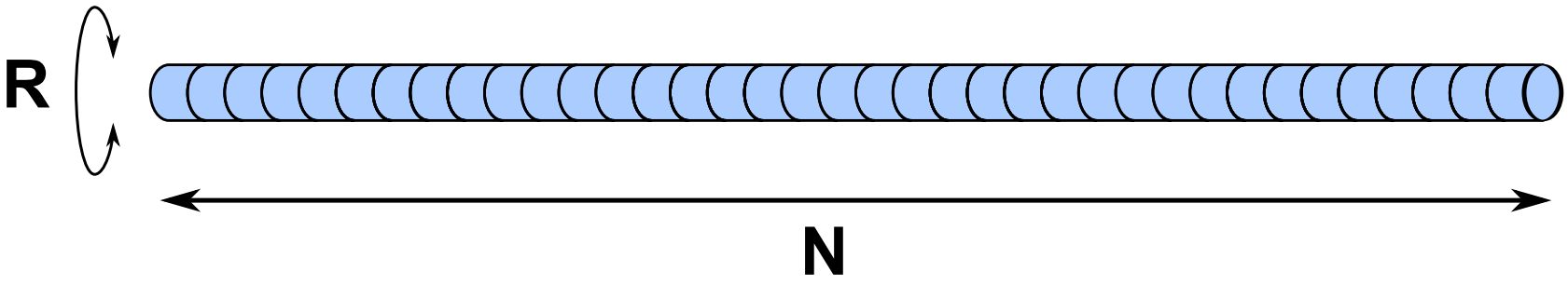


Andraschko and Sirker, PRB 89, 125120 (2014)

Deep quenches for coupled chains

$$H = \sum_i \left\{ H_{1D,i} + J_\perp \int_0^R \mathrm{d}x \; \sigma_i(x)\sigma_{i+1}(x) \right\}$$

As a check, start with small *R*



Chain fermion energies diverge for all but the lowest excited state

$$E_n \sim \sqrt{\Delta^2 + \left(\frac{2\pi n}{R}\right)^2}$$

Deep quenches for coupled chains

$$H = \sum_i \left\{ H_{1D,i} + J_\perp \int_0^R \mathrm{d}x \; \sigma_i(x)\sigma_{i+1}(x) \right\}$$

- As a check, start with small *R*
- Only ground state and first excited state on each chain survive, recover 1D TFIM

$$\lim_{R\to 0} H = \sum_i \tilde{h}\sigma_i^z + \tilde{J}_\perp \sigma_i^x \sigma_{i+1}^x$$

$$\tilde{h} = \Delta + E_{\mathrm{RM},0} - E_{\mathrm{NS},0}$$

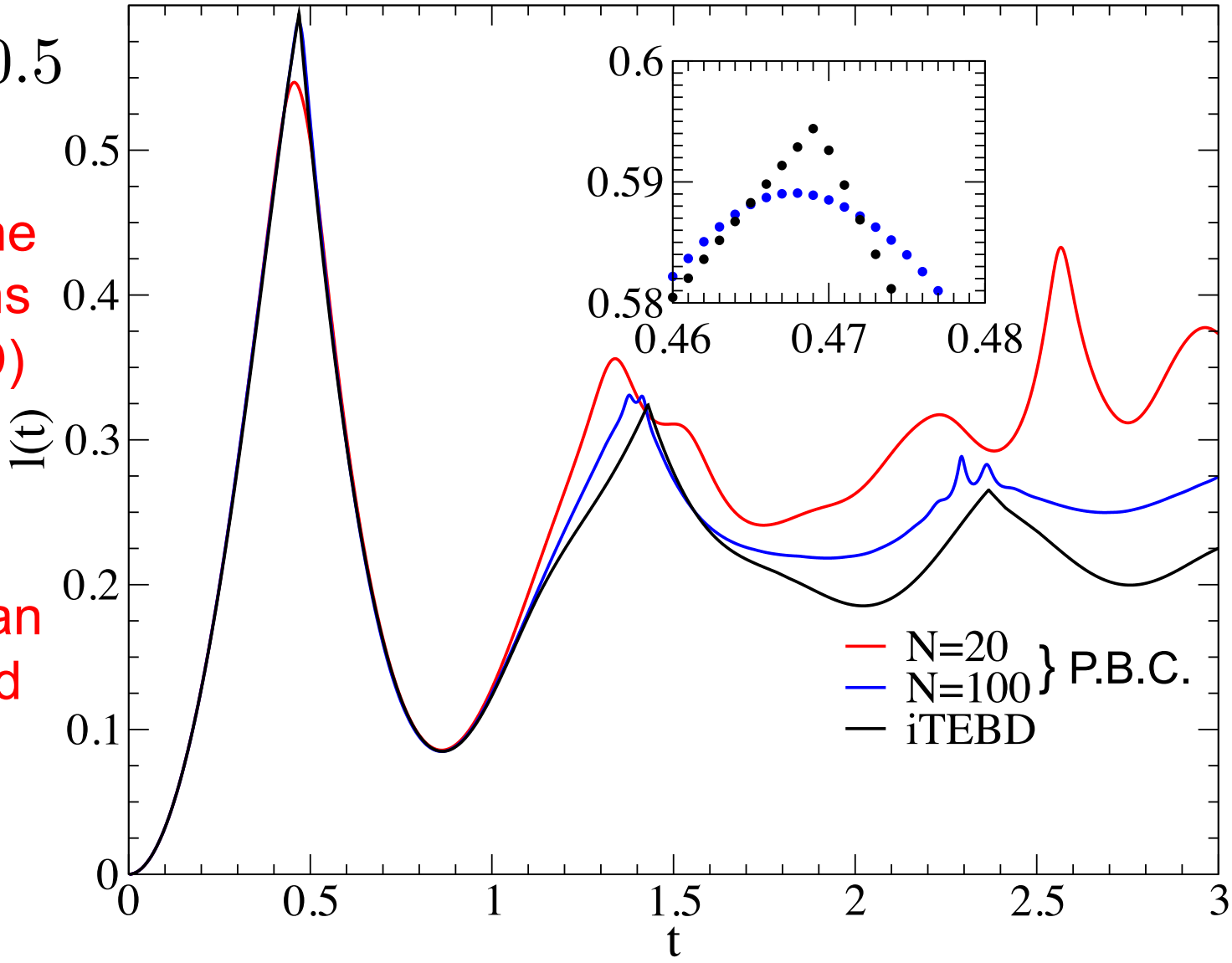
$$\tilde{J}_\perp = J_\perp |\langle \mathrm{RM}, 0|\sigma|\mathrm{NS}, 0\rangle|^2$$

Quench of coupling in TFIM limit $J_\perp = 0$ to $J_\perp = 1.5$

$J_{\perp,crit} \approx 0.5$

Can study the infinite chains case (iTEBD)

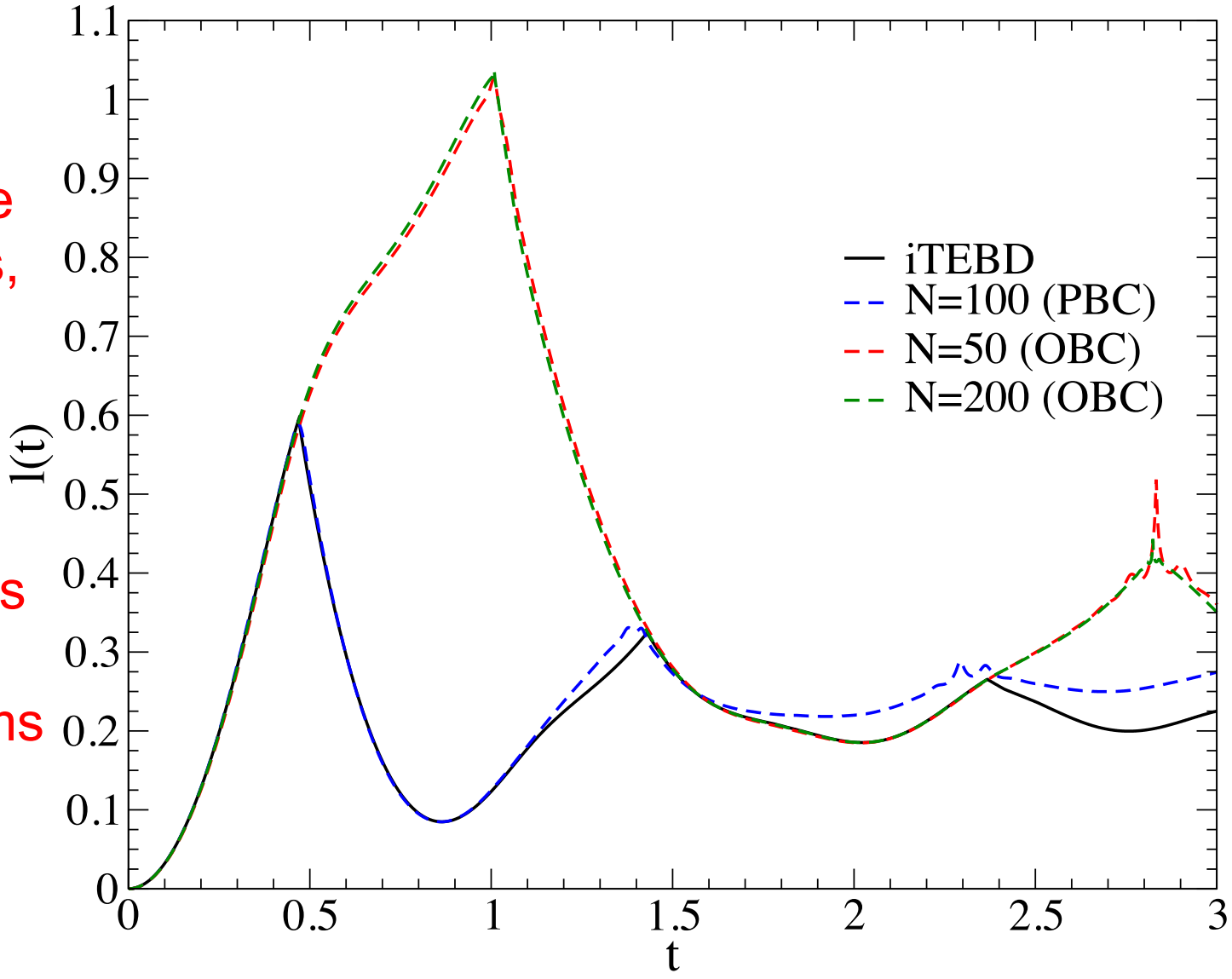
Times for nonanalytic behaviour can be calculated exactly, $t^\star$=0.4693…



Boundary conditions and finite systems

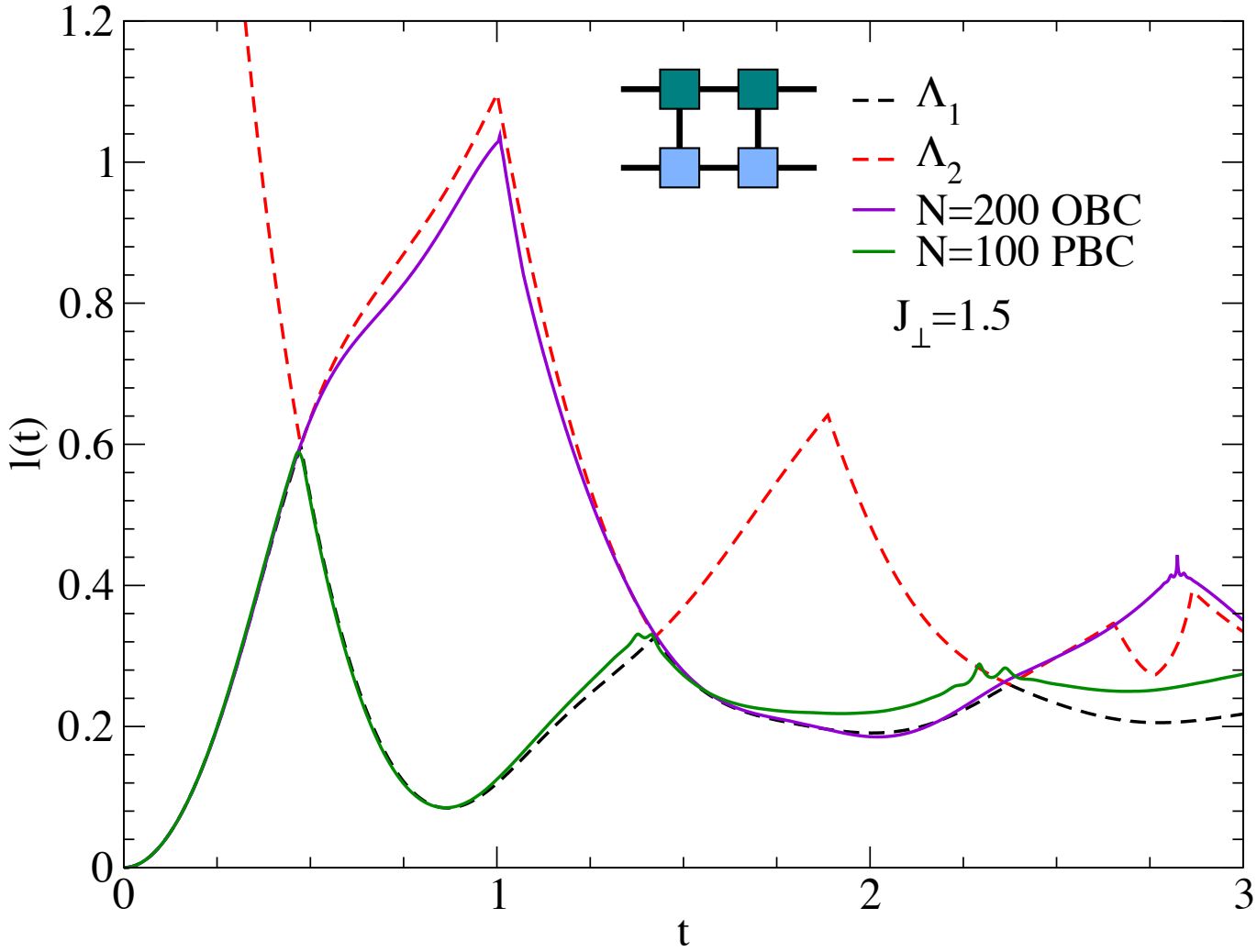
Even for large finite systems, b.c.s make a difference

Important if looking for this behaviour in e.g. cold atoms



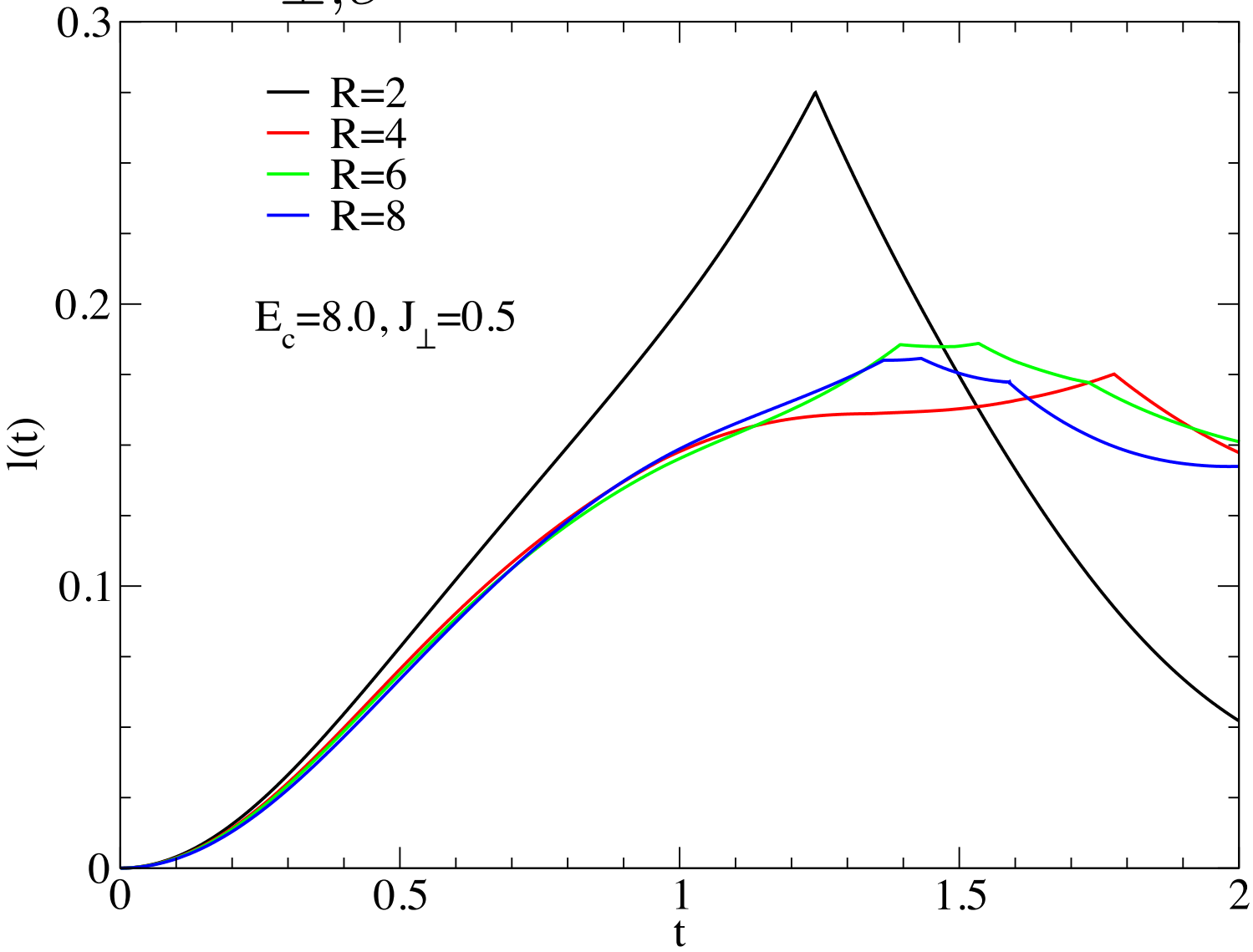
Boundary conditions

Both solutions are visible in the eigenvalues of the iTEBD transfer operator



Deeper quenches for coupled chains

For larger R, $J_{\perp,c} \approx 0.18$



Conclusions

- Coupling integrable chains together allows straightforward extension of 1D MPS methods to 2D systems
  - Extra control parameters are chain length, R, and energy cutoff
  - Can work with infinitely many chains
- Uses
  - Low lying spectrum and phase transitions
  - Investigating entanglement entropy
  - Time evolution, great for shallow quantum quenches, deeper quenches possible for short times

Future work

- Other integrable chains
  - XXZ with J.S. Caux (Amsterdam)
- iDMRG for studying phase transitions
- Better understanding of energy cutoff
- Finite temperature
- Supercomputing