Phase transitions beyond the Landau-Ginzburg theory

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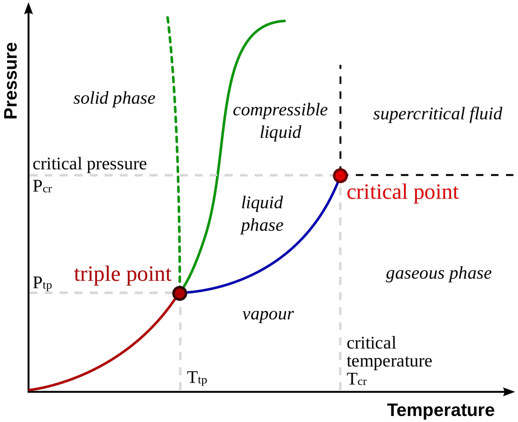
Phase diagram



Order of Phase Transition

- First Order transition: System jumps from one phase to another, at transition point, different phases coexist
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- First Order transition: System jumps from one phase to another, at transition point, different phases coexist
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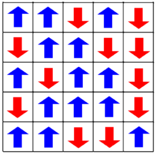
- Free energy F as a function of T: $F(T)$, F is continuous at transition temperature.
  For first order transition $\frac{d}{dT}F$ is discontinuous; second order continuous

Ising model

Let's look at a theoretical model: Ising model.

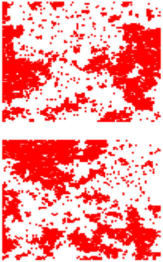


$$H = -J \sum_{<i,j>} S_i S_j$$

- Low T: Spins point to the same direction(Ordered phase)
- High T: Spins point to random direction(Disordered phase)
- Transition point?

Critical point

At critical point, the system looks the same on any scale.
No length scale available!
Up is a typicle
configuration at Critical point.
Down is a "Renormalization"
after grouping 9 spins
together. They look the same!



Scaling and Critical exponents

Near the critical point: $t = (T - T_c)/T_c$

- Correlations: $\langle O_x O_y \rangle \sim |x - y|^{-(d-2+\eta)}$
- specific heat: $C_H \sim |t|^{-\alpha}$
- magnetization: $M \sim |t|^{\beta}$
- magnetic susceptibility: $\chi \sim |t|^{-\gamma}$
- correlation length: $\xi \sim |t|^{-\mu}$

Renormalization Group for 1D Ising model

Dimensionless H:

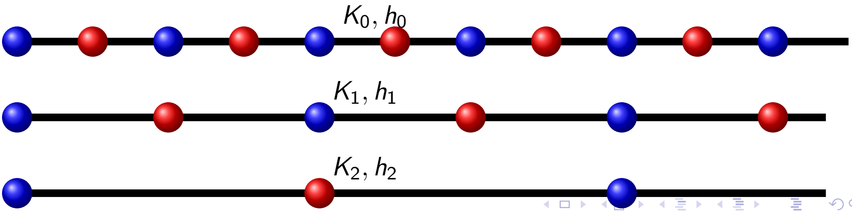
$$H_0 = -\beta H = K_0 \sum_j \sigma_j \sigma_{j+1} + h_0 \sum_j \sigma_j$$

$$Z(N, K_0, h_0) = \sum_{\sigma_1=\pm 1} \sum_{\sigma_2=\pm 1} \cdots \sum_{\sigma_3=\pm 1} exp(K_0 \sum_j \sigma_j \sigma_{j+1} + h_0 \sum_j \sigma_j)$$

Only sum over odd sites:

$$Z(N, K_0, h_0) = g(K_0, h_0) Z(N/2, K_1, h_1)$$

$$Z(N/2, K_1, h_1) = g(K_1, h_1) Z(N/4, K_2, h_2) = \cdot\cdot$$



Renormalization Group Equation

Relations:

$$K_{l+1} = R_K(K_l, h_l)$$

$$h_{l+1} = R_h(K_l, h_l)$$

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Free energy:

$$f(K_0, h_0) = -log(g(K_0, h_0)) + \frac{1}{2} f(K_1, h_1)$$

g is a non-singular function

Order Parameter

For second order phase transition: Order parameter μ:

- μ is 0 in disordered phase
- μ is non-zero in ordered phase
- Free energy:

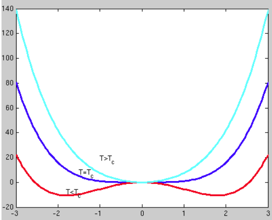
$$F(T,\mu) = F_0 + A(T)\mu^2 + B(T)\mu^4 + \cdot\cdot$$

- $A(T) = \frac{T-T_C}{T_C} + \cdot\cdot, B(T) = B_0 + B_1(T - T_C) + \cdot\cdot$

Minimize the Free Energy:

$$\frac{\partial}{\partial \mu} f = 0$$

$$T > T_C \Rightarrow \mu = 0; \;\; T < T_C \Rightarrow \mu = \pm\sqrt{-\frac{Tc - T}{2B_0}}$$



Calculate critical exponents: Ising model
$\mu = (\frac{Tc - T}{2B_0})^{1/2} \Rightarrow \beta = 1/2$

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Only consider Large distance, low energy behaviors
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- Dimension of system
- Symmetry of Order parameter
- Symmetry and Range of Hamiltonian

XY model in 2D

Superfluid order parameter

Superfluid density: $\Psi(\vec{r}) = |\Psi| e^{i\theta(\vec{r})}$

Only consider the fluctuation in θ, look at the model:

$$H = -J \sum_{<i,j>} cos(\theta_i - \theta_j)$$

Theorem: There's NO spantaneous breaking of continuous symmetry in 2 or less dimensions!

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There's a phase transition nonetheless.

Kosterlitz-Thouless Transition

Vortices as a topological exitation in superfluid

Superfluid order parameter: $\Psi(\vec{r},t) = e^{i\theta(\vec{r},t)}$
Superfluid velocity: $\vec{V}_s(\vec{r},t) = \frac{\hbar}{m}\nabla\theta(\vec{r},t)$

since wavefunction is single valued
phase circulation(change of phase over closed path):
$\oint \nabla\theta \cdot d\vec{l} = 2\pi n \qquad\qquad n = 0, \pm 1, \pm 2, ...$

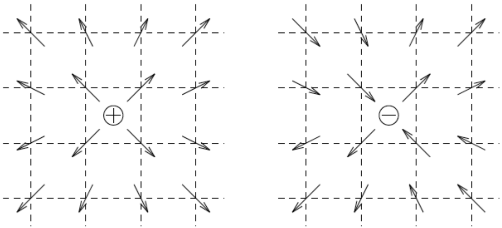


Figure: Vortices with charge ± 1

Kosterlitz-Thouless transition and vortices

Consider in 2D, the energy and entropy of a integer vortex,

$$E = \frac{n_s}{m} \int \mathbf{v}^2 = \frac{\pi n_s \hbar^2}{m} lnL$$

$$S = k_B ln(L)^2$$

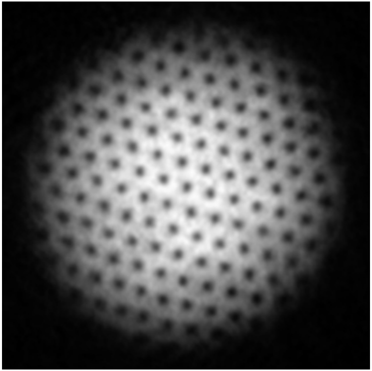
Free energy is:

$$F = U - TS = (\frac{\pi n_s \hbar^2}{m} - 2k_B T) lnL$$

So free vorties emerge when $T_v > \frac{\pi n_s \hbar^2}{2m}$

Kosterlitz-Thouless transition and vortices

- KT transition is related to vortex binding and unbinding
- No magnetization, but chang in the behavior of correlation function



Phase transitions that don't fit the traditional description

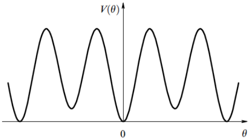
- Topological phase.
  - Doesn't have a local order parameter
  - Depends on the topological properties of the system
- Decomfined transition.
  - We'll see a simple example

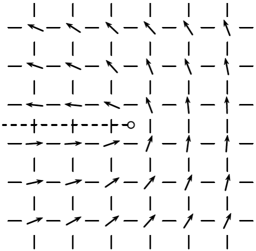
Generalized XY model

Add a term that has π periodicity to the XY Hamiltonian.

$$H = \sum_{<i,j>} [(1-\Delta)cos(\theta_i - \theta_j) + \Delta cos(2(\theta_i - \theta_j))]$$

The extra term enables half vortices!
It can have 1/2 KT transition





Mapping to height model

We fellow Villain model, using this partition function:

$$Z = \int_{-\pi}^{\pi} \prod_{c} \frac{d\theta_c}{2\pi} \prod_{<ab>} \omega(\theta_a - \theta_b)$$

where $\omega(\theta) = e^{-J(1-\Delta)cos(\theta) - J\Delta cos(2\theta)}$
We "cheat" by replacing: $e^{-Jcos(\theta)}$ by
$\omega_V(\theta) = \sum_{p=-\infty}^{+\infty} e^{-\frac{J}{2}(\theta+2\pi p)^2}$ Villain weight.

$$\omega(\theta) = \sum_{p=-\infty}^{+\infty} e^{-\frac{J}{2}(\theta+2\pi p)^2}(1 + (-1)^p e^{-K})$$

Mapping to height modell

We still use Villain model, and consider after a Fourier transform:

$$\omega_V(\theta) = \sum_{n=-\infty}^{+\infty} e^{in\theta} e^{-\frac{T}{2J}n^2}$$

Then we can integrate out the angular dependence, which gives some delta functions,

$$Z = \sum_{n_{i,j},\Delta n=0} exp(-\frac{J_*}{2}\sum_{<i,j>} n_{ij}^2 + \frac{K_*}{2}\sum_{<i,j>}(-1)^{n_{ij}})$$

Where n_{ij} satisfies the current conservation condition, and $J_* = J^{-1}$, $sinhK_* * sinhK = 1$

If we write $n_{ij} = h_i - h_j$, we have the height model:

$$Z = \sum_{h_i} exp(-\frac{J_*}{2}\sum_{<i,j>}(h_i - h_j)^2 + \frac{K_*}{2}\sum_{<i,j>}\mu_i\mu_j)$$

several special limits

$$Z = \sum_{h_i} exp(-\frac{J_*}{2} \sum_{<i,j>} (h_i - h_j)^2 + \frac{K_*}{2} \sum_{<i,j>} \mu_i \mu_j)$$

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- When $K = 0$, the currents can only be even,
  replace h by $h = 2\tilde{h}$, and J by $J/4$ we have the Villain model again,
  we have the critical J is 4 time bigger than Villain model

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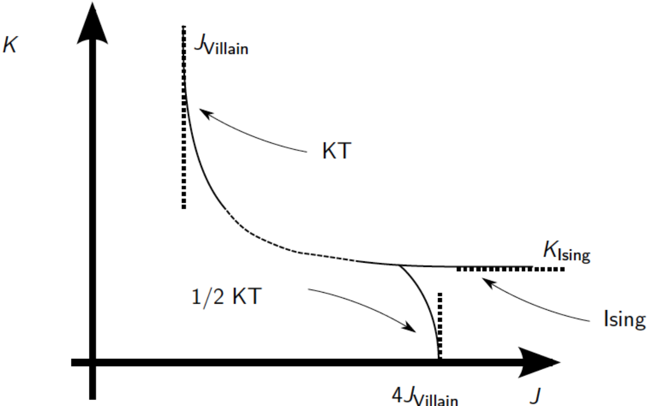
- When $J \to +\infty$, only the even and oddness of the current matters.
  We get an Ising model

Suggested Phase Diagram

It has **KT, 1/2KT** and **Ising** type transitions!
We are most interested in the tricritical region



Continuum limit

Now we can write a continuous field theoretical discription.
$h_i = 2\tilde{h}_i + (\mu_i - 1)/2$, that seperates the Ising and Gaussian parts.

$$\sum_n \delta(\tilde{h} - n) = \sum_q e^{2\pi i q \tilde{h}}$$

keeping only $q = \pm 1, \pm 2$ terms,

$$Z = \sum_{\mu_i = \pm 1} \int \prod_i exp(-2J_* \sum_{<i,j>} (\tilde{h}_i - \tilde{h}_j)^2 + \frac{K_*}{2} \sum_{<i,j>} \mu_i \mu_j$$

$$+ \sum_i z_1 \mu_i sin(2\pi \tilde{h}_i) + \sum_i z_2 cos(4\pi \tilde{h}_i))$$

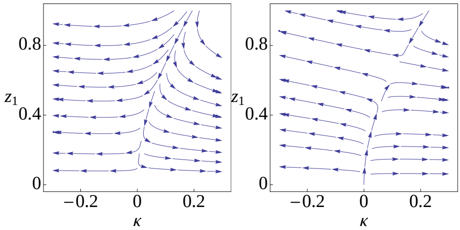
Renormalization

Consider the region that J is just small enough for half vortices, so we can neglect the term that gives integer vortices
and we have the free boson field, Ising term, and a term
$H_{1/2V} = z_1 \int dx \mu(x) sin[2\pi \tilde{h}(x)]$ that couples them.
μ is called the Ising disorder operator
With this term, two half vortices are connected together by an Ising string!

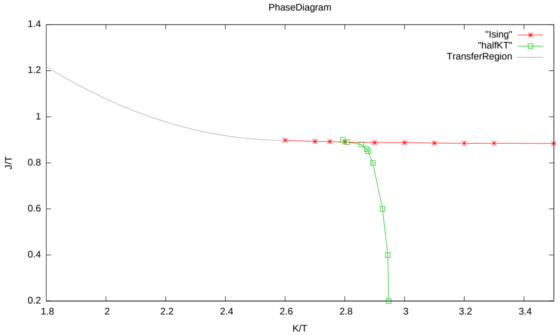
Renormalization

Near the Ising critical point, $cos(2\pi\tilde{h})$ has scaling dimension $\pi J/4$, which is irrelevent when $J < \frac{8}{\pi}$
But $\mu sin(2\pi\tilde{h})$ has scaling dimension $\frac{1}{8} + \frac{\pi J}{4}$, it's relevent untill $J = \frac{15}{2\pi}$ So the Ising transition persists untill after it has met the 1/2KT transition.

Figure: RG flow in fixed J plain



Phase Diagram



Ising transition continues after the 1/2 KT line!

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- Across the Ising transition line, both Ising and KT order parameter gain none-zero value.
- A simple example of decomfined transition in 2D.