Color-Kinematic Symmetry and Gauge-Gravity Connection in the Space of Generalized Propagating Matrix and in Light-Like gauges

York-Peng Edward Yao

University of Michigan, Ann Arbor MI

with Diana Vaman

In the recent past, there have been many *computational advances* in gauge and gravitaional scattering amplitudes. Furthermore, hidden symmetries are found, which connect *color and kinematical labelling* in gauge amplitudes. *Gravitaional and gauge amplitudes are found to be related.*

Symmetries are not just there to organize bookkeeping. In many cases, they are the dynamics.

Our work (so far) has been to set up a natural framework to accomodate these new developments and to enforce these new symmetries. The mystery and the implications of the correspondence and the interplay between dynamics in momentum space and external color symmetries, between gravity and gauge, demand much more explorations.

Much of these advances either originated from or were inspired by string considerations, which are adept at describing amplitudes. However, the dynamical aspects and the gauge degrees of freedoms are succintly summarized in Lagrangians. More importantly, off-shell continuations of amplitudes are well prescribed, once a set of Feynman rules are derived. These will be used.

(1) (long) Introduction

(2) Space-cone Gauge and Analytic Continuation

(3) Color-Kinematic Duality and its Enforcement

(4) Double Copy Connection

Our discussion is mostly for tree amplitudes only.

1. Introduction:

(a) New Techniques in Computation:

The old way (Feynman diagrams + Lorentz covariant rules) is tedious and wasteful. Each diagram yields many terms and then there are tremendous amounts of cancellation among diagrams. Instead, helicity formalism uses the massless spinor solutions for each massless particle p_i

$$\sigma \cdot p_i u_\pm(p_i) = 0,$$

$$< i| = \bar{u}_-(p_i), \quad [i| = \bar{u}_+(p_i),$$

$$|i >= u_+(p_i), \quad |i] = u_-(p_i),$$

$$\epsilon(p_i, q_i)^\mu_+ = \frac{< q_i|\sigma^\mu|p_i\ ]}{\sqrt{2} < q_i\ p_i >},$$

$$\epsilon(p_i, q_i)^\mu_- = \frac{[\ q_i|\tilde{\sigma}^\mu|p_i >}{-\sqrt{2}[q_i\ p_i]},$$

where q_i is a reference null vector. Changing q_i corresponds to a change of gauge $\epsilon_\mu \to \epsilon_\mu + p_\mu \lambda$ and therefore different q_i can be chosen for different particle i.

The amplitudes should be independent of q_i. **By judicious choice of these reference vectors** and some spinor tricks (Chinese tricks), one can simplify the calculation for amplitudes drastically. Some of the results look very simple and elegant, when written in spinors (twistors).

The next major advance is to compute gluon amplitudes recursively. This is due to Witten and his then students (BCFW). They introduced a complex number z into certain momenta and then continued the n particle amplitude

$$A_n \to A_n(z), \quad A_n = A_n(z=0).$$

If $\mathbf{A_n(z \to \infty) = 0}$, we have

$$\oint \frac{dz}{z} A_n(z) = 0.$$

For tree amplitudes, $A_n(z)$ is a rational function of z. The poles (z_m) in z, if coming only from internal propagators, correspond to cutting diagrams into two halves, then **a n particle amplitude is split into a sum of products of two amplitudes with smaller numbers of external particles**

$$A_n = -\sum_{z_m \neq 0} A_m A_{n+2-m} \quad (BCFW\ recursion).$$

z is introduced by shifting the momenta of some external particles Besides having to respect momentum conservation, one must preserve the masslessness of these particles.

The easiest way to satisfy the last requirement is to make the shifts on the spinors,

$$|i>\rightarrow |i> + z|...>, \quad [j| \rightarrow [j| + z[***|$$

because if we write p_i in the form of an outer product

$$\sigma \cdot p_i = |i>[i| \rightarrow det|\sigma \cdot p_i| = 0 = p_i^2.$$

In light-like gauges there are paths in momentum space in which the interaction vertices have no z dependence and therefore the asymptotic condition is trivially satisfied. However, to make shifts in spinors, one should use a subclass space-cone gauges, in which the shifts are z multiplied by the gauge fixing spinors.

(b) Hidden Symmetries in Gauge Amplitudes:

If we follow from left to right, in clockwise or counterclock-wise sense, each Feynman diagram is endowed with a sequence of color indices. For example, for a four gluon process, taking 1 as the starting point, we have (1234), (1243), (1324), (1342), (1423), (1432). However, the gluon couplings are

$$c_i = \sum_b f_{a_1 a_2 b} f_{a_3 a_4 b},$$

which means (i) the four indices split into two pairs with the two indices in each pair being antisymmetric. This reduces the count to (12;34), (13;24), (14;23). When we add all diagrams with the same color coupling together, we get a color-ordered amplitude A_i^4.

The complete amplitude is

$$
\begin{aligned}
A^4 &= \sum_i c_i A_i^4 \\
&= \frac{c_{(12;34)} n(12;34)}{s_{12}} + \frac{c_{(13;24)} n(13;24)}{s_{13}} \\
&\quad + \frac{c_{(14;23)} n_(14;23)}{s_{14}}.
\end{aligned}
$$

The totally antisymmetric f_{abc} satify an identity $\sum_b (f_{a_1 a_2 b} f_{a_3 a_4 b} + f_{a_2 a_3 b} f_{a_1 a_4 b} + f_{a_3 a_1 b} f_{a_2 a_4 b}\ )$ $= 0$, or

$$c_{(14;23)} = c_{(13;24)} - c_{(12;34)},$$

and surprisingly calculating in any gauge one gets

$$n(14;23) = n(13;24) - n(12;34),$$

This means a color-kinematic matching of indices and that there are *at most* two independent color ordered amplitudes, given by

$$|A >= M^{(4)}|N >,$$

$$|A >= \left(\begin{array}{c} A(1234) \\ A(1324) \end{array} \right), \quad |N >= \left(\begin{array}{c} n(12;34) \\ n(13;24) \end{array} \right),$$

and $M^{(4)}$ is a real symmetric generalized propagator matrix

$$M^{(4)} = \left(\begin{array}{cc} \frac{1}{s_{12}} + \frac{1}{s_{14}} & -\frac{1}{s_{14}} \\ -\frac{1}{s_{14}} & \frac{1}{s_{13}} + \frac{1}{s_{14}} \end{array} \right).$$

There is a further surprise, in that $M^{(4)}$ has an eigenvector with zero eigenvalue

$$M^{(4)}|\lambda^0>= 0, \quad |\lambda^0>= \begin{pmatrix} -s_{12} \\ s_{13} \end{pmatrix},$$

which means we can shift the numerators $|N>\rightarrow |N> + f|\lambda^0>$ without changing $|A>$. This is called a *generalized gauge transformation.* If we take $f = -n(13;24)/s_{13}$, we have

$$|N> + f|\lambda^0>= \begin{pmatrix} n(12;34) + \frac{s_{12}}{s_{13}} n(13;24) \\ 0 \end{pmatrix},$$

which gives $A(1234) = \frac{s_{13}}{s_{12}} A(1324)$, or there is only one independent color ordered amplitude for n=4.

Null eigenvector(s) of M reduce the number of independent color-ordered amplitudes. The amplitudes are not changed under any shift of the form $\sum f_i \lambda_i$.

For gauge amplitudes with more than four particles, direct perturbation calculations do not yield numerators n_i which satisfy Jacobi identities. We shall see that nevertheless:

(i) The deviations can be absorbed by a shift in the numerators, such that the new set of numerators $\bar{n}_i$ respect the Jacobi identities without changing the values of the amplitudes A_i. For n particles, Jacobi identities give a set of color coefficients relations

$$c_i + c_j + c_k = 0$$

i, j, k are color ordered labels. The shifted numerators for the same color labels satisfy

$$\bar{n}_i + \bar{n}_j + \bar{n}_k = 0,$$

which is called *color-kinematic duality.*

(ii). If a pair i and j among n external particles share the same vertex, their color factor $f_{a_i a_j b} = -f_{a_j a_i b}$. We find $n(\ldots;ij;\ldots) = -n(\ldots;ji;\ldots)$. With the Jacobi identities, there are $(n-2)!$ independent c_i, n_i. Thus, there can be at most $(n-2)!$ color-ordered amplitudes given by

$$|A >= M^{(n)}|\bar{N} > .$$

$M^{(n)}$ is a real symmetric matrix, which has $(n-3)(n-3)!$ eigenvectors with zero eigenvalue. Corresponding, there are the same number of arbitrary functions which we can attach to

$$|\bar{N} > \rightarrow |\bar{N} > + \sum f_i|\lambda_i^0 >$$

without changing $|A>$. The number of independent A_i is $(n-3)!$.

A straightfoward calculation of n_i in general does not respect color- kinematic duality. The deviations from Jacobi identities

$$n_i + n_j + n_k = \Delta_{ijk},$$

can be calculated recursively, by making some $p_i^2 \neq 0$. They are used to determine the individual δn_i in $\bar{n}_i = n_i + \delta n_i$, such that $\bar{n}_i + \bar{n}_j + \bar{n}_k = 0$, through an equation

$$|D[\Delta] >= M^n |\delta N > .$$

Clearly, these δn_i cannot be reached by generalized gauge transformations. If we summarize their effects on amplitudes in coordinate space in the form of effective vertices, we find that the sum of all these effective vertices vanishes ($L_{eff}(x) = 0$) in view of color Jacobi identities $c_i + c_j + c_k = 0$! We have a connection between color space c_i and momentum space δn_i, via coordinate space L_{eff}.

(c) Connection between gauge theories and gravity:

One exciting motivation to study (BCJ) color-kinematic duality is the observation by the same set of authors that once such a set of dual symmetric numerators is found, then up to a ratio of coupling constants, the n graviton scattering amplitude is given by (Double Copy)

$$A^n_{gr} =< \bar{N}^T | M^{(n)} | \bar{N} >,$$

where gluons and gravitons have the same helicity assignments. This is a remarkable result, which does not just make graviton calculations much easier. It allows us to tackle the renormalizability of gravity (and its extensions) in a new way.

Renormalizability has to do with high energy behavior. We know that of $|\bar{N} >$. What about the combinations in the double copy formula

$$A^n_{gr} = \sum < \lambda_\alpha | \bar{N} >^2 (\frac{1}{\lambda})_\alpha,$$

Pure gravity is finite at one loop, which implies much better behavior from the combinations.

2. Space-cone gauge and BCFW:

Both gluons and gravitons have only two helicity states. One reason that earlier calculations were tedious is because to maintain covariance, we worked with many more components. However, these physical systems, because of unitarity, know that the unphysical degrees of freedom are superfluous and must cancel, which happen at an alarming rate. The BCFW method is a clear indication that all that matters are the physical states. Therefore, at least for tree level consideration, we are better off using physical gauges to amplify the real issues..

A set of guages with this demand is the light-like gauges. It is characterize by a light-like vector N, such that for gauge theories $N_\mu A^\mu_a = 0,\ \ N^2 = 0.$

Take two light-like vectors

$$\sigma \cdot N_- = |- > [-|, \quad \sigma \cdot N_+ = |+ > [+|,$$

$< +- >= [-+] = 1$, we form two more

$$\sigma \cdot \bar{N} = -|+ > [-|, \quad \sigma \cdot N = -|- > [+|,$$

For a light-like vector P, we write

$$\sigma \cdot P = p^+ N_+ + p^- N_- - pN - \bar{p}\bar{N} = |p > [p|$$

$$p = -2P \cdot \bar{N} =< p+ > [p-],$$
$$\bar{p} = -2P \cdot N =< p- > [p+],$$
$$p^{\pm} = -2P \cdot N_{\mp} =< p\mp > [\mp p|.$$

The scalar product of two light like vectors P, Q $(= -\frac{1}{2} < pq > [qp],)$ is given as

$$P \cdot Q = \frac{1}{2}(p\bar{q} + \bar{p}q - p^{+}q^{-} - p^{-}q^{+}).$$

For general four vectors, we symbolicall decompose into these components and use the same metric for scalar products. The space cone gauge for a gauge field is defined by imposing

$$a_b = 0.$$

$\bar{a}_b$ is then a dependent component. We can express the Lagrangian in terms of $a_b^{\pm}$, which will be identified as $\pm$ helicity fields. After a rescaling of fields by a factor $\sqrt{2}$ so that the free Lagrangian gives the usual $\frac{1}{p^2}$ propagator and $g \to \frac{g}{\sqrt{2}} = 1$ so that $F^{\mu\nu}$ as a whole gains $\sqrt{2}$, we have

$$
\begin{aligned}
L = & - a_a^- \partial_\mu \partial^\mu a_a^+ + f_{bac}(\frac{\partial^+}{\partial} a_b^-) a_a^- \partial a_c^+ \\
& + f_{bac}(\frac{\partial^-}{\partial} a_b^+) a_a^+ \partial a_c^- \\
& - (f_{bac} a_c^- \partial a_a^+) \frac{1}{\partial^2} (f_{bef} a_f^+ \partial a_e^-).
\end{aligned}
$$

We note that there is no $\bar{\partial}$ in the interaction part of this Lagrangian. Therefore, if we make a momentum shift in the $|+>[-|$ direction (multiplied by z), then we would not induce z-dependent terms in the numerators for the amplitude. z appears only in the propagators, except for diagrams with the four particle vertex, which does not always come with propagators. We can choose $N_\pm$ such that the four vertex gives no contribution. Then $A(z \to \infty) = 0$ and we can perform BCFW continuation for recursion.

(3) Color-Kinematic Duality:

The claim (BCJ) is that if the color coefiicients satisfy $c_i + c_j + c_k = 0$, then one can find a set of numerators which also satisfy $n_i + n_j + n_k = 0$, for which the amplitude is

$$A = \sum_i \frac{c_i n_i}{\Pi_j s_j}.$$

Note that c_i and n_i here are the complete set of color coefficients and numerators. The immediate questions are: (a) Is the claim true? and/or (b) How do we make that set? If you apply Feynman rules in the space-cone gauge and calculate the five point amplitudes, you will find that without some massaging,

$$n_i + n_j + n_k = \Delta_{ijk} \neq 0.$$

Let us take a five particle amplitude. By distributing the four point vertices appropriately to n_i (multiplying by $1 = s_j/s_j$), we find that we have a set of 15 numerators. We can write them in the form of

$$n(ij;k;lm) = -n(ji;k;lm) = -n(ij;k;ml) \\ = -n(ml;k;ji),$$

The color coefficients satisfy the Jacobi identities

$$c(ij;k;lm) + c(ki;j;lm) + c(jk;i;lm) = 0,$$

but not the n's. By applying Feynman rule calculation, we obtan

$$n(ij;k;lm)+n(ki;j;lm) + n(jk;i;lm) \\ = \Delta(ijk/lm),$$

We make a shift in n_i, i. e.

$$n_i \to \bar{n}_i + \delta n_i,$$

demanding

$$\bar{n}(ij;k;lm) + \bar{n}(ki;j;lm) + \bar{n}(jk;i;lm) = 0,$$

and

$$A = \sum_i \frac{c_i \bar{n}_i}{\Pi_j s_j} \to \sum_i \frac{c_i \delta n_i}{\Pi_j s_j} = 0.$$

$$\delta n(ij;k;lm) + \delta n(ki;j;lm) + \delta n(jk;i;lm) \\ = -\Delta(ijk/lm),$$

These conditions show that there are only 6 independent $\bar{n}_i$, δn_i and result in two sets of equations for the independent color amplitudes and δn_i.

$$\begin{pmatrix} A(12345) \\ A(14325) \\ A(13425) \\ A(12435) \\ A(14235) \\ A(13245) \end{pmatrix} = M^{(5)} \begin{pmatrix} \bar{n}(12345) \\ \bar{n}(14325) \\ \bar{n}(13425) \\ \bar{n}(12435) \\ \bar{n}(14235) \\ \bar{n}(13245) \end{pmatrix},$$

and

$$\begin{pmatrix} D(12345) \\ D(14325) \\ D(13425) \\ D(12435) \\ D(14235) \\ D(13245) \end{pmatrix} = M^{(5)} \begin{pmatrix} \delta n(12345) \\ \delta n(14325) \\ \delta n(13425) \\ \delta n(12435) \\ \delta n(14235) \\ \delta n(13245) \end{pmatrix},$$

$D_i's$ are linear in $\Delta(ijk/lm)$. The easiest way to obtain $\Delta(ijk/lm)$ is first to obtain off-shell $\Delta(ijk/l)$. This recursive construction works for any number of particles.

$M^{(5)}$ is a generalized propagator matrix, It is symmetric and has four null eigenvectors. We use them to perform a $\sum g_i|\lambda_i^0>$ shift and reduce $|\delta N>$ to having only two non-zero elements. For the helicity $1^+2^-3^+4^-5^+$ they are given, due to the left hand side (D's), as

$$\delta n' = s_{12}\frac{s_{45}}{s_{24}}X, \quad \delta n'' = -s_{25}\frac{s_{14}}{s_{24}}X,$$

where

$$\begin{aligned} X =& \frac{p_{\bar{1}}}{p_1}(p_{52} - p_{54}) + \frac{p_{\bar{5}}}{p_5}(p_{12} - p_{14}) \\ &- \frac{p_{\bar{3}}}{p_3}(p_{12} - p_{14} + p_{52} - p_{54}), \end{aligned}$$

$$p_{i2} = \frac{p_i p_2}{p_i + p_4}, \quad p_{i4} = \frac{p_i p_4}{p_i + p_2}.$$

While $\delta n'$ and $\delta n''$ are linear combinations of the six independent δn_i, these two equations still allow us to solve for all the δn_i, if we insist that δn_i should satisfy the natural symmetry under the interchange of the indices $1 \leftrightarrow 5$ and/or $2 \leftrightarrow 4$

$$2 \leftrightarrow 4 \quad \delta n(12;3;45) \leftrightarrow -\delta n(14;3;52), \ etc.$$

We assume that all the δn_i have the structure

$$\begin{aligned}
&\frac{p_1^-}{p_1}\left(x^{32}p_{32} + x^{34}p_{34} + x^{52}p_{52} + x^{54}p_{54}\right)\\
&+ \frac{p_3^-}{p_3}\left(y^{12}p_{12} + y^{14}p_{14} + y^{52}p_{52} + y^{54}p_{54}\right)\\
&+ \frac{p_5^-}{p_5}\left(z^{12}p_{12} + z^{14}p_{14} + z^{32}p_{32} + z^{34}p_{34}\right),
\end{aligned}$$

and generate a set of linear imhomogeneous equations for x, y, z.

We solve these equations and obtain the independent δn_i and then use the Δ_{ijk} for the dependents ones also. The total effect on the scattering amplitude can be summarized by an effective Lagrangian:

$$\begin{aligned}
&\Big[(f_{dec_3}f_{c_4c_5e}+f_{dec_4}f_{c_5c_3e}+f_{dec_5}f_{c_3c_4e})\frac{f_{c_2c_1d}}{s_{c_1c_2}}\\
&+(f_{dec_2}f_{c_3c_4e}+f_{dec_3}f_{c_4c_2e}+f_{dec_4}f_{c_2c_3e})\frac{f_{c_5c_1d}}{s_{c_1c_5}}\\
&+(f_{dec_1}f_{c_3c_5e}+f_{dec_3}f_{c_5c_1e}+f_{dec_5}f_{c_1c_3e})\frac{f_{c_4c_2d}}{s_{c_2c_4}}\Big]\\
&\quad\times(\frac{\partial^-}{\partial}a^+_{c_1})(\partial a^-_{c_2})a^+_{c_3}\frac{1}{\partial}(a^-_{c_4}\partial a^+_{c_5})=0.
\end{aligned}$$

We anticipate that it should vansish, because we demand the color ordered amplitudes not to change by the shifts, and we worked hard to obtain these shifts in momentum space. However, the above Lagrangian trivially vanishes because of the Jacobi identity within each pair of round parenthesis! Colors and kinematics and are again intertwined!

(4) Double Copy Connection:

The gravitational Lagrangian is

$$L = -\sqrt{-g} g^{\mu\nu} R_{\mu\nu}, \quad g = det(g_{\mu\nu})$$

Because the theory is invariant under variations of four infinitesimal parameters $\delta\xi^\mu$ in

$$\delta g_{\mu\nu} = -\delta\xi^\gamma \partial_\gamma g_{\mu\nu} - g_{\mu\gamma}\partial_\nu \delta\xi^\gamma - g_{\gamma\nu}\partial_\mu \delta\xi^\gamma$$

we can impose four conditions, which in the light-cone gauge are

$$g^{++} = g^{+i} = 0,$$

and

$$-g_{+-} = k^l, \quad k \equiv det(g_{ij})$$

where l is a real number.

The 'phase' components of g_{ij} are the dynamical degrees of freedom

$$g_{ij} = k^{1/2} e_{ij}, \quad det(e_{ij}) = det(e^{ij}) = 1.$$

(This reminds one of a non-linear sigma model.) There are various ways of parameterizing e_{ij}, such as

$$e_{ij} = \sqrt{1 - det(h)}\delta_{ij} + h_{ij},$$
$$e^{ij} = \sqrt{1 - det(h)}\delta_{ij} - h_{ij},$$

with

$$h_{ij} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}.$$

α, β are the Hermitian fields associated with helicity $+, -$. These various parametrization will give rise to different contact terms in the Lagrangian, when we expand out the square root and the determinant.

For the three point vertices, we find that

$$(1^+2^+3^-)_{gr} \sim ((\frac{p_1}{p_1^-} - \frac{p_2}{p_2^-})\bar{p}_3)^2 \sim (1^+2^+3^-)^2_{gauge}.$$

For the four particle amplitude, we find that there are more than ten four point vertices in α and β. We can find a choice of reference vectors q such that they don't contribute to the amplitudes and the double copy formula follows. It seems, however, that there must be some representation(s) of e_{ij} such that we shall obtain the double copy result independent of choice of reference vectors. The challenge is what is the underlying principle for one to accomplish that. Preferably, it is due to some symmetry consideration.