Full counting statistics and Edgeworth series for Matrix Product State

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21 November

The AKLT state, where everything started

Spin-1 : $H = \sum_{<i,j>} \vec{S}_i \cdot \vec{S}_j + \Delta(\vec{S}_i \cdot \vec{S}_j)^2$
$\Delta = \frac{1}{3}$, Ground state, AKLT state, Affleck, Kennedy, Lieb and Tasaki

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Think of Spin-1 as 2 spin-$\frac{1}{2}$

 $= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

 $= |+\rangle\langle\uparrow\uparrow| + |0\rangle\frac{\langle\uparrow\downarrow| + \langle\downarrow\uparrow|}{\sqrt{2}} + |-\rangle\langle\downarrow\downarrow|$

Also a simple example of topological state

Gappless edge modes, string order parameter, fractional charge

AKLT state, the explicit form

Use the physical picture to write the state:

$$A^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

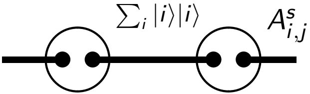
$$A^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$|AKLT\rangle \propto \sum_{\{s_i\}} \mathrm{Tr}(\prod_i A^{s_1} A^{s_2} \ldots A^{s_n}) |s_1 s_2 \ldots s_n\rangle$$

Generalization of AKLT state to Matrix Product State
Physical intuition

- Physical Space: d Auxiliary Space: DxD
- Neighboring auxiliary spins in a maximumly entangled state: $\sum_{i=1}^{D} |i\rangle|i\rangle$



- Use matrix $A^{s_k}_{i,j}$ to project from $D \times D$ dimensional auxiliary space to d dimensional physical space

Generalization of AKLT state to Matrix Product State
Mathematical forms

- $$|\Psi_M\rangle = \sum_{\{s_i\}} \prod_i \mathrm{Tr}(A_{[1]}^{s_1} A_{[2]}^{s_2} ... A_{[N]}^{s_N}) |s_1, s_2, ..., s_N\rangle (\mathrm{PBC})$$
- $$|\Psi_M\rangle = \sum_{\{s_i\}} \prod_i A_{[1]}^{s_1} A_{[2]}^{s_2} ... A_{[N]}^{s_N} |s_1, s_2, ..., s_N\rangle (\mathrm{OBC})$$

  $A_1^{s_i}$ and $A_N^{s_i}$ $1 \times D$ and $D \times 1$ dimensional
- Variational ansatz with $D \times D \times d \times N$ numbers.
- $D \sim e^N$, every state is exactly a MPS
  $D = 1$ meanfield limit
  D finite, only access a very small portion of Hilbert space!

Schmidt decomposition

Divide system into subsystem A(L sites) B($N-L$ sites)

$$|\psi\rangle = \sum_{i=1}^{d^L} \sum_{j=1}^{d^{N-L}} C_{ij} |i\rangle_A \otimes |j\rangle_B$$

Single value decomposition: $C = UDV$, U and V unitary, D diagonal with semipositive elements called Schmidt coefficients λ_α

$$|\psi\rangle = \sum_{\alpha=1}^{\chi} \lambda_\alpha \Big(\sum_{i=1}^{d^L} U_{i\alpha} |i\rangle_A\Big) \otimes \Big(\sum_{j=1}^{d^{N-L}} V_{\alpha j} |j\rangle_B\Big)$$
$$= \sum_{\alpha=1}^{\chi} \lambda_\alpha |\phi_\alpha^{[A]}\rangle \otimes |\phi_\alpha^{[B]}\rangle$$

$\langle \phi_\beta^{[A]} | \phi_\alpha^{[A]} \rangle = \delta_{\alpha\beta}$ and $\langle \phi_\beta^{[B]} | \phi_\alpha^{[B]} \rangle = \delta_{\alpha\beta}$

Schmidt decomposition
Reduced density matrix and entanglement

$$\rho_A = \sum_{\alpha=1}^{\chi} \lambda_\alpha^2 |\phi_\alpha^{[A]}\rangle\langle\phi_\alpha^{[A]}|$$

$$\rho_B = \sum_{\alpha=1}^{\chi} \lambda_\alpha^2 |\phi_\alpha^{[B]}\rangle\langle\phi_\alpha^{[B]}|$$

$$S_{[A,B]} = -\sum_{\alpha=1}^{\chi} \lambda_\alpha^2 \log(\lambda_\alpha)^2$$

rank of $\rho_{A,B}$ = rank of D

Schmidt decomposition and Canonical form

- Gauge freedom, $A_{[i]} \to X_i^{-1} A_{[i]} X_{i+1}$, state invariant. Divide system into two parts,

$$|\Psi_M\rangle = \sum_{\alpha=1}^{D} |\phi_\alpha^{left}\rangle \otimes |\phi_\alpha^{right}\rangle$$

$$\phi_\alpha^{left} = \sum_{\{s_1,\ldots s_i\}} A_{[1]}^{s_1} A_{[2]}^{s_2} \ldots A_{[i]}^{s_i} |s_1, s_2, \ldots, s_i\rangle$$

$$\phi_\alpha^{right} = \sum_{\{s_{i+1},\ldots s_N\}} A_{[i+1]}^{s_{i+1}} A_{[i+2]}^{s_{i+2}} \ldots A_{[N]}^{s_i} |s_{i+1}, s_{i+2}, \ldots, s_N\rangle$$

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- Canonical Form: one particular gauge choice, so that the above equation is the Schmidt composition

Properties of MPS

ρ_L Reduce density matrix of L neighboring spins
rank of $\rho_L \leq D$ (D^2 for PBC)

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rank of $\rho_L \leq D$ (D^2 for PBC)
Tranlational invariant case: $|\Psi_M\rangle = \sum_{\{s_i\}} \mathrm{Tr}(\prod_i A_{s_i})|\{s_i\}\rangle$

- Normalization:

$$|\Psi_M|^2 = \mathrm{Tr}[(E_A)^N]$$

$E_A = \sum_i A_{s_i} \otimes \bar{A}_{s_i}$

- Operator expectation value:

$$\langle \hat{O}_i \rangle = \mathrm{Tr}((E_A)^{N-1} E_A^O)$$

$$\langle \hat{O}_i \hat{O}'_j \rangle = \mathrm{Tr}(E_A^O (E_A)^{j-i-1} E_A^{O'} (E_A)^{N-j+i-2})$$

$E_A^O = \sum_{i,j} O_{i,j} A_{s_i} \otimes \bar{A}_{s_j}$

Two point function

Consider two point function: $C(r) = \langle \hat{O}_0 \hat{O}_r \rangle - \langle O \rangle^2$

$$|\Psi_M|^2 = \mathrm{Tr}[(E_A)^N] \to \lambda_M^N$$

$$\langle \hat{O}_i \rangle \to (l|E_A^O|r)$$

λ_M largest eigenvalue, set to 1, $(l|$ and $|r)$ eigenvectors $(l|r) = 1$
Second largest eigenvalue $\lambda_2 < 1$, $\xi = \log(1/\lambda_2)$:

$$C(r) \propto exp(-l/\xi)$$

Second largest eigenvalue $= 1$:

$$C(r) \to \mathrm{const}$$

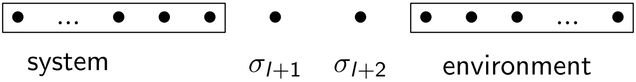
Properties of MPS

- Two point function: $\langle \hat{O}_0 \hat{O}_r \rangle - \langle O \rangle^2$ either decays exponentially or stays as constant
  MPS cannot describe critical system!
  In practice, one will have an effective correlation length $\xi \propto \log D$
- Matrix Product State represent ground state faithfully
  F. Verstraete and J. I. Cirac, Phys. Rev. B 73, 094423 (2006)

MPS and DMRG

Density Matrix Renormalization Group
Calculate ground state energy to almost machine precision

- Variational method with MPS ansatz
- Fix all A_{s_i} except on one site, minimize the energy, and move to the nex site



DMRG and truncation

Truncation is essential

DMRG and truncation

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Truncate

Only keep D states

1D gapped system, the eigenvalues of ρ_L decay exponentially*

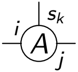
Not true in 2D!

Most error comes from truncation

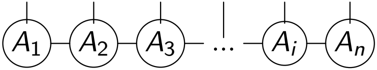
* U. Schollwck RevModPhys.77.259

Graphic representation

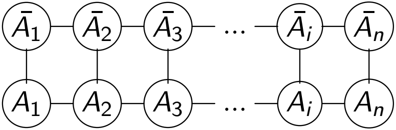
$A^{s_k}_{i,j}$:



Ψ_M :

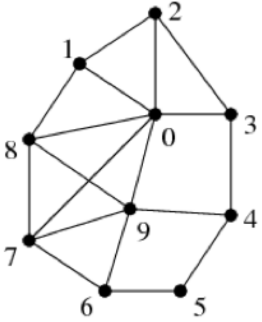


Calculate normalization:



Tensor Network State

Create any Tensor Network State



Entanglement is build in!

Tensor Network State

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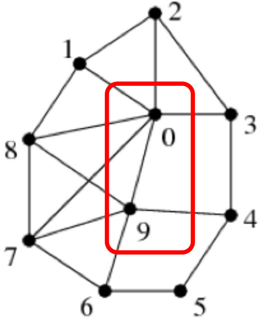
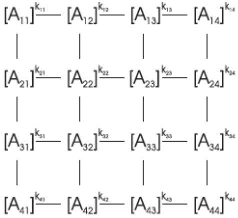


Entanglement is build in! $S \sim \#$ of legs cut by the partition

PEPS

Projected Entanglement Pair State

Straight forward generalization to 2D



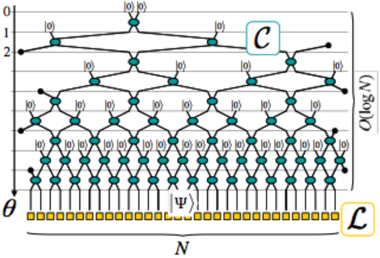
Can also describe critical system

Impossible to compute anything! (normalization, operator expectation value...)

F. Verstraete, J.I. Cirac, V. Murg, arXiv:0907.2796

MERA

Multi-scale entanglement renormalization ansatz
Designed to describe Critical system, for block of L sites,
$S_E \sim \log(L)$



Found its role in AdS/CFT

G.Vidal, Phys.Rev.Lett.101.110501

Full counting statistics

FCS generating function:

$$\chi(\lambda) = \sum_n P_n e^{i\lambda n}$$
$$log\chi(\lambda) = \sum_n \frac{\kappa_n (i\lambda)^n}{n!}$$

κ cumulants, $\kappa_1 = \langle n \rangle$, $\kappa_2 = \langle n^2 \rangle - \langle n \rangle^2$ ...
It characterizes quantum noise and fluctuation*.
Related to entanglement entropy †

*L. S. Levitov and G. B. Lesovik, JETP Lett. 58, 230 (1993)
†I. Klich and L. Levitov, Phys. Rev. Lett. 102, 100502 (2009)

Full counting statistics

An example

Consider Poisson distribution:

$$P(n) = \frac{\alpha^n e^{-\alpha}}{n!}$$

$$\chi(\lambda) = \sum_{n=0}^{\infty} \frac{\alpha^n e^{-\alpha}}{n!} e^{i\lambda n} = e^{-\alpha(1-e^{i\lambda})}$$

$$log(\chi(\lambda)) = \alpha(i\lambda + \frac{(i\lambda)^2}{2} + \frac{(i\lambda)^3}{6} ...)$$

$\langle \hat{J} \rangle = e^* \langle \hat{n} \rangle$, $\langle \hat{J}^2 \rangle = e^{*2} \langle \hat{n}^2 \rangle$ So that,

$$\frac{\langle \hat{J}^2 \rangle - \langle \hat{J} \rangle^2}{\langle \hat{J} \rangle^2} = e^*$$

FCS for MPS

Assume translational invariance, infinitely long chain
Prob distribution of $\sum S_z$:

$$\chi(\lambda; l) = \sum_n P(S_z = n) e^{i\lambda n}$$
$$= \frac{\mathrm{Tr}(\mathrm{E}_A(\lambda)^l \mathrm{E}_A(0)^{N-l})}{\mathrm{Tr} E_A(0)^N} \sim \chi_0(\lambda)\chi_1(\lambda)^l$$

Bulk term and boundary term
A central limit theorem!
"normalize" the variable, $S_z \to \frac{S_z - \mu}{\sigma}$, Gaussian distribution
MPSs are finitely correlated

Edgeworth expansion

Correction to Central Limit Theorem?

For IID, Edgeworth series: $\chi_M(\lambda; l) = (1 + \sum_{j=1}^{\infty} \frac{q_j(i\lambda)}{l^{j/2}})e^{-\lambda^2/2}$

So that:

$$F_l(x) = \Phi(x) + \sum_{j=1}^{\infty} \frac{q_j(-\partial_x)}{l^{j/2}} \Phi(x)$$

$q_1 = \frac{1}{6}\kappa_3(i\lambda^3)$
$q_2 = \frac{1}{24}\kappa_4(i\lambda^4) + \frac{1}{72}\kappa_3^2(i\lambda)^6$
...
$\Phi(x)$ error function, κ_i is the ith cumulant

Edgeworth seires for MPS

$$\ln(\chi_0(\lambda)) = \sum_{r=1}^{\infty} \frac{\xi^r(\lambda)^r}{n!}$$

$$\ln(\chi_1(\lambda)) = \sum_{r=1}^{\infty} \frac{\kappa^r(\lambda)^r}{n!}$$

Normalize distribution:

$$\hat{M}_l = \frac{1}{\sqrt{l}} \frac{\hat{S}_l - l\mu(l)}{var(\sigma, l)}$$

$$\ln\chi_M(\lambda; l) = \frac{(i\lambda)^2}{2} + \frac{(i\lambda)^3(l\kappa_3 + \xi_3)}{6(l\kappa_2 + \xi_2)^{3/2}} + \frac{(i\lambda)^4(l\kappa_4 + \xi_4)}{24(l\kappa_2 + \xi_2)^2} + \frac{(i\lambda)^5(l\kappa_5 + \xi_5)}{120(l\kappa_2 + \xi_2)^{5/2}} + \dots$$

Edgeworth seires for MPS

$$F_l(x) = \Phi(x) + \sum_{j=1}^{\infty} \frac{q_j(-\partial_x)}{l^{j/2}} \Phi(x)$$

First few terms:

$$q_1 = -\frac{\kappa_3 (\partial_x)^3}{6\kappa_2^{3/2}}$$

$$q_2 = \frac{\kappa_4 (\partial_x)^4}{24\kappa_2^2} + \frac{\kappa_3^2 (\partial_x)^6}{72\kappa_2^3}$$

$$q_3 = -\frac{\kappa_3^3 (\partial_x)^9}{1296\kappa_2^{9/2}} - \frac{\kappa_3 \kappa_4 (\partial_x)^7}{144\kappa_2^{7/2}} - \frac{\kappa_5 (\partial_x)^5}{120\kappa_2^{5/2}} - \frac{(\partial_x)^3}{6}(\xi_2 - \frac{3\xi_2}{2\kappa_2})$$

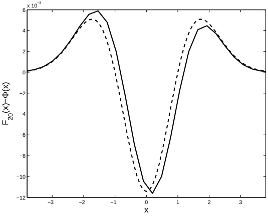
Pseudo-probability distribution

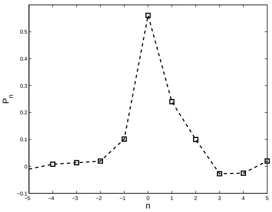
For MPS: $\chi_1(\lambda)$ pseudo-probability distribution.
Not real probability distribution, but:

- χ_1 is periodic so distribution is discrete
- Fourier component of χ_1 is real
- Fourier components sum to 1

Example

$$A^+ = \sqrt{\frac{1}{3}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}; A^0 = \sqrt{\frac{1}{6}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}; A^- = \sqrt{\frac{1}{3}}\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}$$





The topological case

AKLT state

- Diffrent behavior of FCS

$$E(\lambda) = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 2e^{\mathrm{i}\lambda} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2e^{\mathrm{i}\lambda} & 0 & 0 & 1 \end{pmatrix}$$

$\chi_1 = 1$ because of topological order

- Only edge freedom can fluctuate!

Conclusion

- Matrix product state powerfull in 1D
- Full counting statistics for MPS reveals the nature of the state
- Topological properties can also be shown from FCS