Hydrodynamic Noise and Bjorken Expansion

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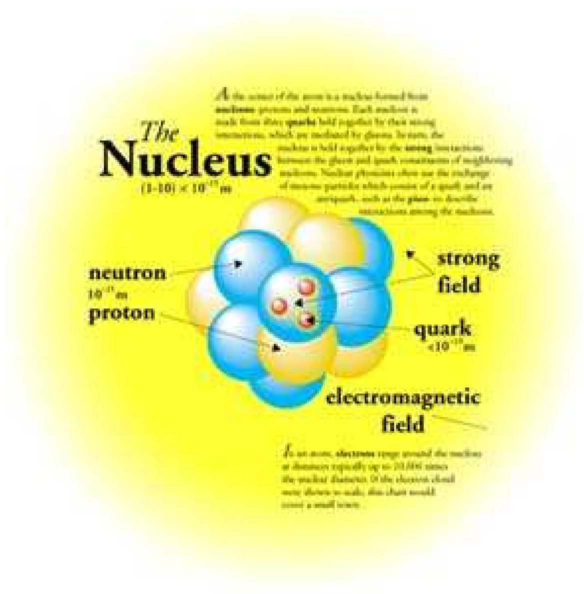
Strong interactions



- **Nuclear force** holds protons/neutrons together in a nucleus.

  This is the force that makes the Sun shine.

- Like van-der-Waals forces in ED, nuclear force is a “shadow” of a much stronger force.

  This force holds proton’s constituents together.

- Like in ED the strong force field can carry waves – gluons.
- Unlike ED the gluon-mediated force grows with separation. Confinement.
  The “charge” comes in 3 varieties – colors.
  The QFT of the gluons and quarks – **Quantum Chromodynamics**.
- QCD is the most elegant piece of the Standard Model.
  The quest to “solve” QCD produced the most advanced ideas and tools in Theoretical Physics – from Lattice to String theory.

Quark-Gluon Plasma

- Can the gas of color-neutral nucleons be “ionized”?

  Can the (confined) quarks be set free by heating the gas of hadrons to extremely high temperature, revealing the QCD constituents and the “color” forces?

  Quark-Gluon Plasma.

- What temperature would be needed?
  $k_B T = \hbar c / R_{\rm proton} \sim 200$ MeV (over 10^5 of T in the Sun’s core)

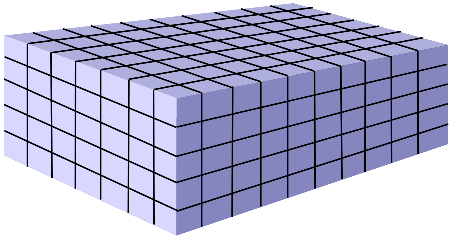
- The Universe was that hot (and hotter) at the beginning.

- Today we can recreate such conditions by smashing large atomic nuclei.

RHIC 

LHC

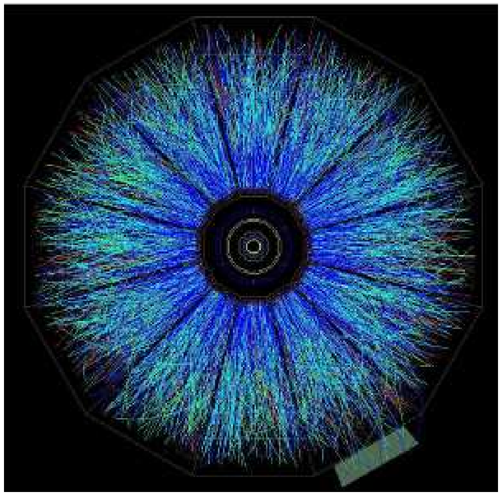
Lattice

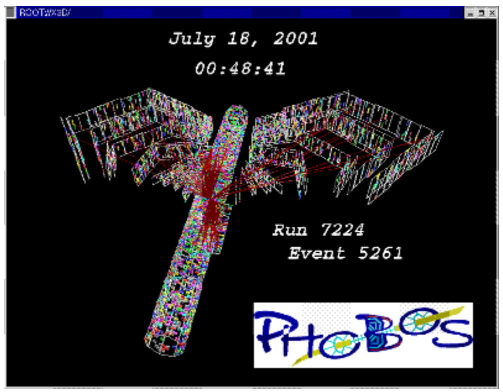


- QCD is a quantum field theory, i.e., everything measurable is, in principle, predictable: hadron masses, scattering amplitudes, equation of state, etc.

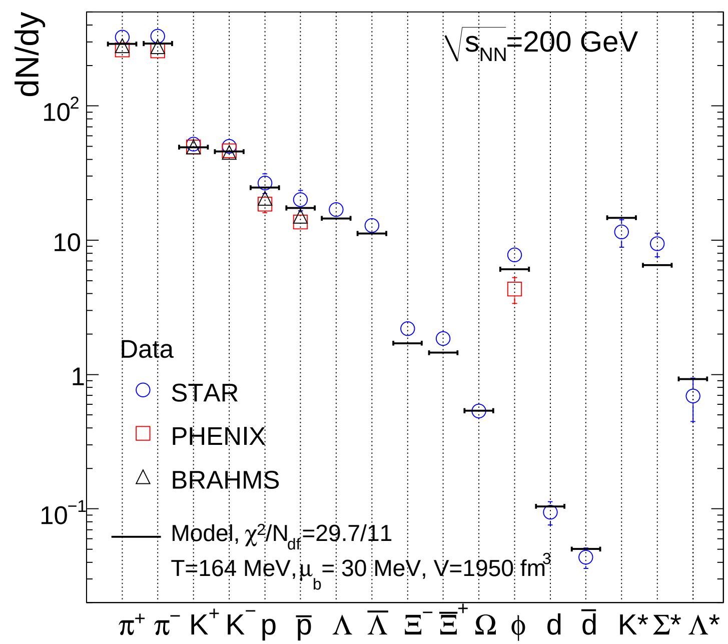
- The theory says "calculate path integral" (Feynman), i.e., a weighted sum over all space-time trajectories of the variables, i.e., the fields.
- It is an infinitely difficult problem (for a non-trivial theory), because the number of variables (and integrations) is ∞.
- Lattice approach allows to reach this infinity gradually, by starting on a coarse, finite lattice and refining it. QCD on the lattice (Wilson).
- The question the lattice can answer is: what happens in QCD at finite T in equilibrium?
  - E.g., how does energy density, pressure, entropy, etc. depend on T?
  - Does hardron gas become QGP at $T \sim 200$ MeV? Yes, it does.
- But non-equilibrium (e.g., transport) properties are still a challenge for today's lattice methods.

Heavy-ion collision





- Heavy-ion collision creates matter in not quite as static a state as what we can study on the lattice.
- The created fireball evolves (explodes): "little bang". Transport properties are important.
- Detectors measure the particle type and momentum distributions in the final state, when the density drops so that the particles free-stream – freeze-out.



- This state is thermal, with temperature about 160 MeV.

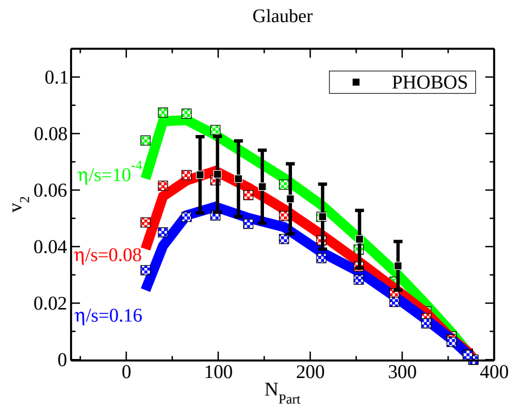
  Similarity to BB and CMB.

Hydrodynamic description

- The hydrodynamic description of the heavy ion collision goes back to Landau (1953).

Approach: take the equation of state, set initial conditions, and solve hydrodynamic equations to get particle yields, spectra, etc.

- Good agreement with data.
- Sensitive to viscosity. (Azimuthal asymmetry)



- Recent interest is due to the remarkably small implied value of viscosity.

  Expressed as the ratio η/s it is much smaller than one would predict in a weakly-coupled QCD plasma (Arnold-Moore-Yaffe, 2003).

  The ratio η/s is a measure of the coupling strength.

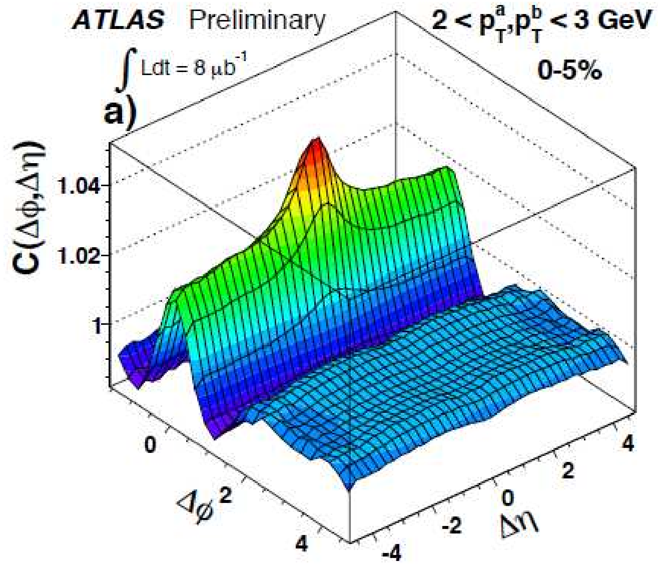
  For weak coupling $\dfrac{\eta}{s} \sim \dfrac{1}{(\text{coupling})^2}$ – must be large.

- AdS/CFT calculation in SYM theory at <u>infinite</u> coupling: $\eta/s = 1/(4\pi)$.

- QGP at RHIC is not a gas, but a very good (perfect?) liquid. sQGP.

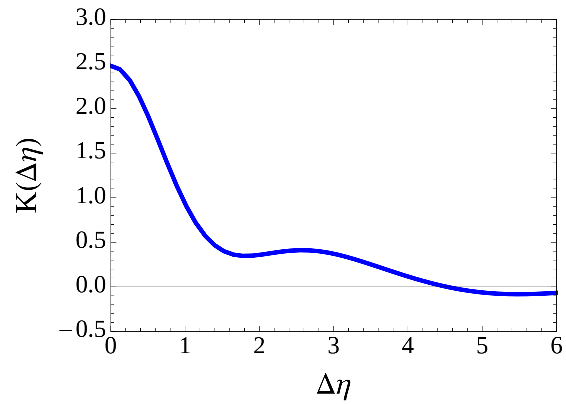
Fluctuations and viscosity

- Can viscosity be measured in a different, complementary way?
- Idea: fluctuation-dissipation theorem requires fluctuations (hydrodynamic noise) and dissipation (viscosity) to be proportional to each other.
- The magnitude of fluctuations (correlations) can be measured. ☞



- The hydrodynamic correlations can be determined theoretically and depend on viscosity.
- Correlations over large $\Delta\eta$ are induced by local fluctuations (hydrodynamic noise) propagating with the speed of sound.

  Similarity to the fluctuations in the CMB.



Relativistic Hydrodynamics

Hydrodynamics: the effective theory for slow, long scale variations of the variables characterizing local thermal equilibrium.

Variables: conserved quantities – energy, momentum, charge densities.

Equations: conservation laws – $\nabla_\mu T^{\mu\nu} = 0$, $\nabla_\mu J^\mu = 0$.

Defining variables involves chosing the local rest frame: $T^{00} = \epsilon$, $J^0 = n$.

Simplifying choice(s): $T^{0i} = 0$ (Landau) or $J^i = 0$ (Eckart).

Use 3 components of u^μ instead of momentum density.

The 4 equations involve 10 components of $T^{\mu\nu}$. Thus the remaning 6 must be expressed in terms of ϵ and u^μ.

In equilibrium the medium is homogeneous and ($T^{00} = \epsilon$, $T^{11} = p$, . . .)

$$T^{\mu\nu}_{\rm eq} = \epsilon u^\mu u^\nu - P(\epsilon)(g^{\mu\nu} - u^\mu u^\nu).$$

$\nabla_\mu T^{\mu\nu}_{\rm eq} = 0$ is ideal hydrodynamics.

Viscous hydrodynamics

Deviations from equilibrium are due to (slow) spatial variations of ϵ and u^μ. I.e.,

$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \underbrace{\text{(gradients of } \epsilon \text{ and } u^\mu\text{)}}_{\Delta T^{\mu\nu}}$$

Subject to $\Delta T^{\mu\nu} u_\nu = 0$ ($T^{\mu\nu} u_\nu = \epsilon u^\mu$ by definition), the most general form is ($\Delta^\mu = h^{\mu\nu}\nabla_\nu$, $h^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$)

$$\Delta T^{\mu\nu} = \eta \left[\Delta^\mu u^\nu + \Delta^\nu u^\mu - \frac{2}{3} h^{\mu\nu} (\nabla \cdot u) \right] - \zeta h^{\mu\nu} (\nabla \cdot u)$$

Second law of thermodynamics for entropy $s = \beta(\epsilon + P)$ flow

$$\nabla_\mu (s u^\mu) = -\beta \Delta T^{\mu\nu} \nabla_\mu u_\nu = \frac{\eta}{2T} \left[\Delta^\mu u^\nu + \Delta^\nu u^\mu - \frac{2}{3} h^{\mu\nu} (\nabla \cdot u) \right]^2 + \frac{\zeta}{T} (\nabla \cdot u)^2$$

Fluctuations and Noise

So far hydro. eqns. describe evolution of the average values of the variables. In a thermodynamic ensemble the variables fluctuate.

The origin of the noise is local, but these fluctuations propagate according to hydrodynamic equations. I.e., hydrodynamics can describe long-range correlations.

This means

$$\Delta T^{\mu\nu} = \Delta T^{\mu\nu}_{\rm visc} + S^{\mu\nu}\,.$$

Locality means $\left\langle\, S^{\mu\nu}(x) S^{\alpha\beta}(0) \,\right\rangle \sim \delta^4(x)$. The magnitude is determined by the condition that the equilibrium distribution is given by e^S (Einstein).

Fluctuations and Noise

Generically, for a system of many variables x_i, obeying

$$\dot{x}_i = -\sum_j \gamma_{ij} X_j + y_i$$

where $X_j = -\partial S/\partial x_i$, the required noise is $\langle\, y_i(t) y_j(0)\,\rangle = (\gamma_{ij} + \gamma_{ji})\delta(t)$.

Applying to $x_i \sim \epsilon, u^\mu$ one finds

$$\left\langle\, S^{\mu\nu}(x) S^{\alpha\beta}(0)\,\right\rangle = 2T\left[\eta\left(h^{\mu\alpha}h^{\nu\beta} + h^{\mu\beta}h^{\nu\alpha}\right) + \left(\zeta - \frac{2}{3}\eta\right) h^{\mu\nu}h^{\alpha\beta}\right]\delta^4(x)$$

This, with $T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \Delta T^{\mu\nu}_{\rm visc} + S^{\mu\nu}$ defines a system of stochastic equations $\nabla_\mu T^{\mu\nu} = 0$.

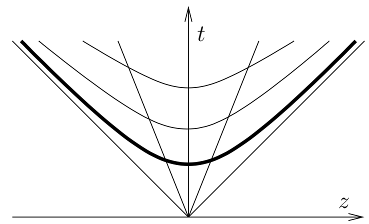
The correlation functions of $T^{\mu\nu}$ can be now calculated by solving in terms of $S^{\mu\nu}$.

Usually, this is applied to fluctuations around a static equilibrium solution. Our goal is to apply this to determine correlations in an expanding fireball.

Bjorken expansion

Bjorken (1983) suggested that the central region of the heavy-ion collisions can be described by a solution of the hydrodynamic equations which is boost-invariant.

The Bjorken flow is conveniently viewed in Bjorken coordinates:



$$t = \tau \cosh \xi \quad \text{and} \quad z = \tau \sinh \xi.$$

In these coordinates the fluid is at rest locally: $u^\mu = (1, 0, \mathbf{0}_\perp)_{\mathrm{Bj}}$.

The average quantities depend only on τ. But the fluctuations depend also on ξ and $\boldsymbol{x}_\perp$, e.g., $\epsilon = \epsilon_0(\tau) + \delta\epsilon(\tau, \xi, \boldsymbol{x}_\perp)$.

We integrate (average) over $\boldsymbol{x}_\perp$ and consider, effectively, a 1+1 dimensional problem. In this case $S^{\mu\nu} u_\nu = 0$ means

$$S^{\mu\nu} = w(\tau) f(\xi, \tau) h^{\mu\nu}$$

where f is random noise ($w = \epsilon + p$)

$$\langle\, f(\xi_1, \tau_1) f(\xi_2, \tau_2)\, \rangle = \frac{2T(\tau_1)}{A\tau_1 w^2(\tau_1)} \left[\frac{4}{3}\eta(\tau_1) + \zeta(\tau_1)\right] \delta\left(\tau_1 - \tau_2\right) \delta\left(\xi_1 - \xi_2\right)$$

Hydrodynamic equations

The only nontrivial function is $\epsilon(\tau)$, and it obeys

$$\frac{d(\tau s)}{d\tau} = \frac{\nu}{\tau T} s$$

I.e., entropy per unit rapidity, $\tau s A$, increases only due to viscosity.

Convenient notation: $\nu \equiv (4\eta/3 + \zeta)/s$.

E.g., for $s \sim T^3 \Rightarrow T \sim \tau^{-1/3} +$ visc. corrections

Hydrodynamic equations for fluctuations

- Fluctuations, $\epsilon = \epsilon_0(\tau) + \delta\epsilon(\xi,\tau)$, $u^z = \sinh(\xi + \omega(\xi,\tau))$, obey

$$\tau \frac{\partial \delta\epsilon}{\partial \tau} + \delta w + w f - \frac{\delta(\nu s)}{\tau} + \left[w - 2\frac{\nu s}{\tau} \right] \frac{\partial \omega}{\partial \xi} = 0$$

$$\tau \frac{\partial}{\partial \tau} \left[\omega \left(w - \frac{\nu s}{\tau} \right) \right] + 2\omega \left(w - \frac{\nu s}{\tau} \right) + \frac{\partial}{\partial \xi} \left[\delta P + w f - \frac{\delta(\nu s)}{\tau} \right] - \frac{\nu s}{\tau} \frac{\partial^2 \omega}{\partial \xi^2} = 0\,.$$

  Easy to obtain by $P \to P + wf - \frac{\nu s}{\tau}\left(1 + \frac{\partial \omega}{\partial \xi}\right)$ in ideal equations.

- Convenient variable: $\rho \equiv \delta s/s = \delta\epsilon/w$. Also, Fourier transform $\xi \to\ k$.

  Solve for $X = \rho, \omega$ in terms of f.

$$\tilde{X}(k,\tau) = -\int_{\tau_0}^{\tau} \frac{d\tau'}{\tau'} \tilde{G}_X(k;\tau,\tau') \tilde{f}(k,\tau')$$

  Then calculate correlations $\langle\, XY \,\rangle$ using known $\langle\, f f \,\rangle$.

Correlations

The equal-(proper)time correlation function at the freeze-out time $\tau_{\rm f}$:

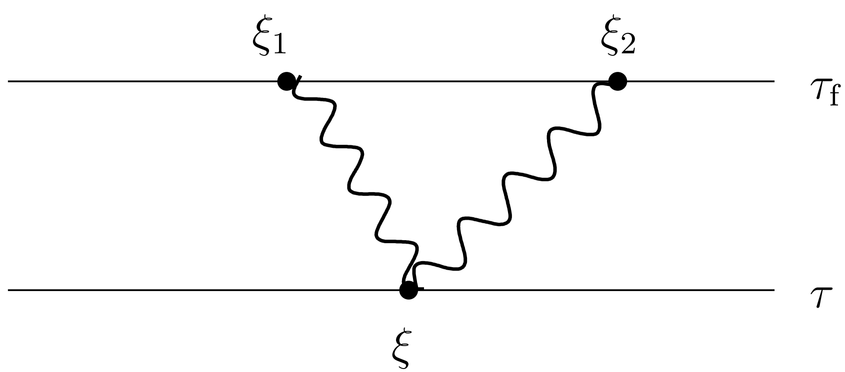
$$C_{XY}(\xi_1-\xi_2;\tau_{\rm f}) \equiv \langle\, X(\xi_1,\tau_{\rm f})\, Y(\xi_2,\tau_{\rm f})\,\rangle = \frac{2}{A}\int\limits_{\tau_0}^{\tau_{\rm f}} \frac{d\tau}{\tau^3}\,\frac{\nu(\tau)}{w(\tau)}\, G_{XY}(\xi_1-\xi_2;\tau_{\rm f},\tau)\,, \quad (1)$$

where the Fourier transform of $G_{XY}(\xi;\tau_{\rm f},\tau)$ is given by

$$\tilde G_{XY}(k;\tau_{\rm f},\tau) \equiv \tilde G_X(k;\tau_{\rm f},\tau)\tilde G_Y(-k;\tau_{\rm f},\tau)\,. \quad (2)$$

Thus

$$G_{XY}(\xi_1-\xi_2;\tau_{\rm f},\tau) = \int_{-\infty}^{\infty} d\xi G_X(\xi_1-\xi;\tau_{\rm f},\tau)G_Y(\xi_2-\xi;\tau_{\rm f},\tau)\,. \quad (3)$$



Solution: inviscid case, linear EOS

$$\tau \frac{\partial \tilde{\boldsymbol{\psi}}}{\partial \tau} + \boldsymbol{D} \tilde{\boldsymbol{\psi}} + \tilde{\boldsymbol{n}} = 0,$$

with (for inviscid case)

$$\tilde{\boldsymbol{\psi}} = \begin{pmatrix} \tilde{\rho} \\ \tilde{\omega} \end{pmatrix}; \quad \boldsymbol{D} = \boldsymbol{D}_0 \equiv \begin{pmatrix} 0 & ik \\ ik{v_{\rm s}}^2 & 1 - {v_{\rm s}}^2 \end{pmatrix}, \qquad \tilde{\boldsymbol{n}} = \begin{pmatrix} 1 \\ ik \end{pmatrix} \tilde{f}\,.$$

$$\tilde{\boldsymbol{\psi}}(k,\tau) = -\int_{\tau_0}^{\tau} \frac{d\tau'}{\tau'} \tilde{\boldsymbol{U}}(k;\tau,\tau') \tilde{\boldsymbol{n}}(k,\tau')$$

where

$$\tilde{\boldsymbol{U}}(k;\tau,\tau') = \mathcal{T} \exp\left\{ -\int_{\tau'}^{\tau} \frac{d\tau''}{\tau''} \boldsymbol{D}(k,\tau'') \right\}$$

If ${v_{\rm s}}^2 \equiv dP/d\epsilon = \text{const}$ (linear EOS):

$$\tilde{\boldsymbol{U}}(k;\tau,\tau') = \frac{(\tau'/\tau)^{\lambda_-}}{\lambda_+ - \lambda_-} \begin{pmatrix} \lambda_+ & -ik \\ -ik{v_{\rm s}}^2 & -\lambda_- \end{pmatrix} - \frac{(\tau'/\tau)^{\lambda_+}}{\lambda_+ - \lambda_-} \begin{pmatrix} \lambda_- & -ik \\ -ik{v_{\rm s}}^2 & -\lambda_+ \end{pmatrix}.$$

$$\boldsymbol{D} \tilde{\boldsymbol{\psi}}_\pm = \lambda_\pm \tilde{\boldsymbol{\psi}}_\pm : \quad \lambda_\pm = \alpha \pm \beta; \quad \alpha = \frac{1}{2}\left(1 - {v_{\rm s}}^2\right); \quad \beta = \sqrt{\alpha^2 - {v_{\rm s}}^2 k^2}.$$

Response functions and sound horizon

$$\tilde{G}_\rho(k;\tau,\tau') = \left(\frac{\tau'}{\tau}\right)^\alpha \left[\cosh\left(\beta\ln(\tau/\tau')\right) + \left(\frac{\alpha+k^2}{\beta}\right)\sinh\left(\beta\ln(\tau/\tau')\right)\right]$$

Note: $\beta = \sqrt{\alpha^2 - {v_{\rm s}}^2 k^2}$ is pure imaginary if $|k| > (1 - {v_{\rm s}}^2)/2v_{\rm s}$.

Acoustic oscillations.

- G_X – meromorphic function of k. Sole singularity is at $k = \infty$.
  Cauchy theorem gives:

  $$G_X(\xi;\tau,\tau') = 0 \quad \text{when} \quad |\xi| > v_{\rm s}\ln(\tau/\tau') \qquad \text{— sound horizon}$$

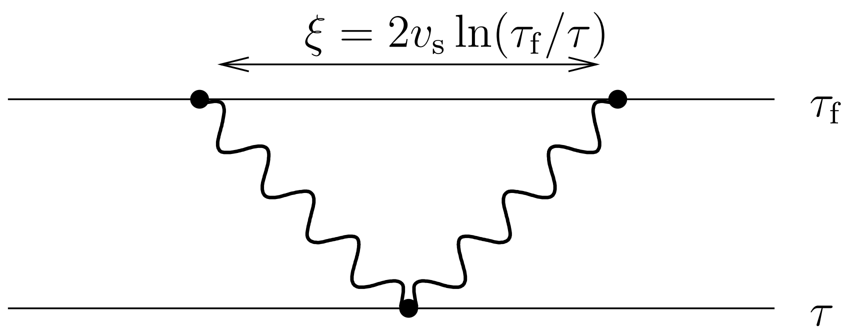
  In the local rest frame the velocity of the front, $\tau d\xi/d\tau$, equals $v_{\rm s}$.

Singularities at the sound horizon

Oscillatory behavior at large k translates into sound front in ξ:

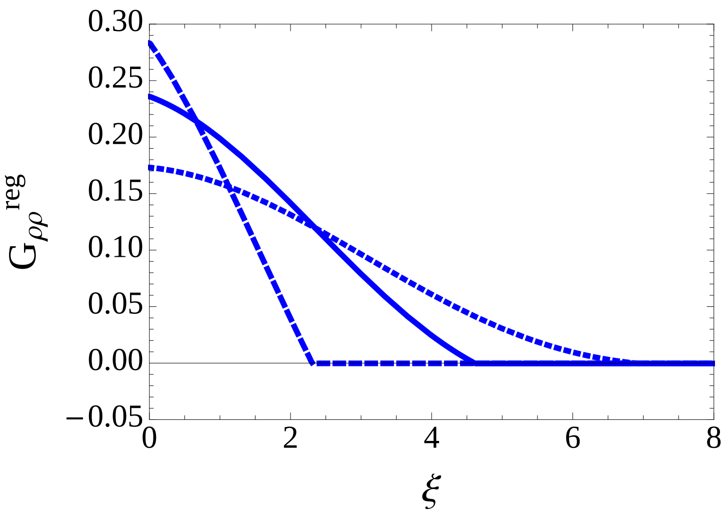
$$G_{\rho\rho}(\xi;\tau_{\rm f},\tau) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}{\rm e}^{ik\xi}\tilde{G}_\rho(k;\tau_{\rm f},\tau)\tilde{G}_\rho(-k;\tau_{\rm f},\tau) \to$$

$$\frac{1}{4{v_{\rm s}}^2}\left(\frac{\tau}{\tau_{\rm f}}\right)^{2\alpha}\left[\delta''(\xi-2v_{\rm s}\ln(\tau_{\rm f}/\tau))+\delta''(\xi+2v_{\rm s}\ln(\tau_{\rm f}/\tau))-2\delta''(\xi)\right]$$

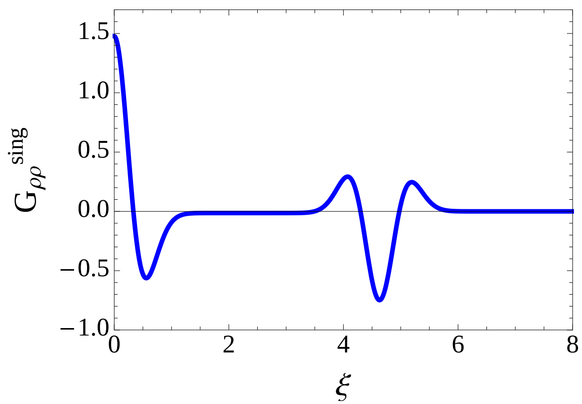


The wake

$$G_{\rho\rho}^{\rm reg} \equiv G_{\rho\rho} - G_{\rho\rho}^{\rm sing}$$



${v_{\rm s}}^2 = 1/3$ and $\ln(\tau_{\rm f}/\tau) = 2, 4, 6$



$\sigma^2 = 0.1$

If the dispersion was linear, there would only be the sound front. However,
$\omega = i\lambda_\pm = i\alpha \pm \sqrt{{v_{\rm s}}^2 k^2 - \alpha^2}$.

Also note that one eigenvalue $\lambda_- \sim k^2$, for $k \to 0$.
Diffusion-like, but no dissipation.

$G_{\rho\rho}$ becomes Gaussian at large $\tau_{\rm f}/\tau$. Width $\langle\, \Delta\xi^2 \,\rangle = 2{v_{\rm s}}^2/\alpha \cdot \ln(\tau_{\rm f}/\tau)$.

Sum rule: $\int_{-\infty}^{\infty} d\xi\, G_{\rho\rho}(\xi;\tau_{\rm f},\tau) = 1$.

Viscosity and taming of singularities

For $\nu = \mathrm{const}$ can be solved by perturbation in $\dfrac{\nu}{\tau T} \ll 1$.

Not assuming $k^2 \times \dfrac{\nu}{\tau T}$ to be small.

$$\tilde{G}_\rho(k;\tau,\tau') = \left(\frac{\tau'}{\tau}\right)^\alpha \left[\cosh\left(\beta \ln(\tau/\tau')\right) + \left(\alpha + k^2 + \frac{\nu k^2}{2\tau' T(\tau')}\right) \frac{\sinh\left(\beta \ln(\tau/\tau')\right)}{\beta}\right] \times \exp\left[-\frac{\nu k^2}{4\alpha}\left(\frac{1}{\tau' T(\tau')} - \frac{1}{\tau T(\tau)}\right)\right]$$

For $G_{XX}(\xi;\tau_{\mathrm{f}},\tau))$ – Gaussian smearing with width

$$\sigma^2 = \frac{\nu}{\alpha}\left(\frac{1}{\tau T} - \frac{1}{\tau_{\mathrm{f}} T_{\mathrm{f}}}\right)$$

Freeze-out

Cooper-Frye prescription:

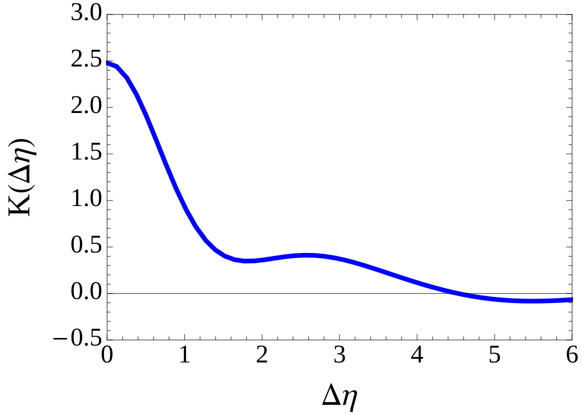
$$p^0 \frac{dN_\mathrm{s}}{d^3p} = d_\mathrm{s} \int_{\Sigma_\mathrm{f}} \mathrm{d}^3\sigma_\mu p^\mu \; f_\mathrm{s}(\boldsymbol{x}, \boldsymbol{p}); \quad f_\mathrm{s}(\mathbf{x}, \mathbf{p}) = \left(\mathrm{e}^{p\cdot u/T} \pm 1\right)^{-1}$$

Fluctuations ρ and ω (i.e., T and u) translate into δN:

$$\delta\left(\frac{dN}{d\eta}\right) = \frac{d_\mathrm{s} A \tau_\mathrm{f} {T_\mathrm{f}}^3}{(2\pi)^2} \int d\xi \; \frac{\rho\, {v_\mathrm{s}}^2 + \omega\; \tanh(\eta-\xi)}{\cosh^2(\eta-\xi)}\, \Gamma\left(4, \frac{m_0}{T_\mathrm{f}} \cosh(\eta-\xi)\right)$$

Translation from ξ to η leads to additional (thermal) smearing.

$$\left\langle \delta\frac{dN}{d\eta_1}\, \delta\frac{dN}{d\eta_2} \right\rangle \left\langle \frac{dN}{d\eta} \right\rangle^{-1} = \frac{45 d_\mathrm{s}}{4\pi^4 N_\mathrm{eff}(T_0)} \, \frac{\nu}{T_\mathrm{f}\tau_\mathrm{f}} \left(\frac{T_0^2}{{T_\mathrm{f}}^2}\right)^{{v_\mathrm{s}}^{-2}-2} K(\Delta\eta)\,,$$



Conclusions and Outlook

- Hydrodynamics predicts long range correlations induced by thermal noise.
- Wake: magnitude is proportional to ν. Nontrivial consequence of expansion. Magnitude of correlations in static equilibrium is determined by the static (thermodynamic) quantities only.
- Ridge: need to detemine ϕ-dependence. Expect a narrow (thermal) peak.
- More realistic (numerical) calculations need to be compared with experiment.