Torsion and Topological Insulators

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with Taylor Hughes and Eduardo Fradkin.

Topological Insulators

- in condensed matter theory, there has been a lot of interest in states of matter known generically as topological insulators (and topological superconductors)
- these are of interest because they do not fall into the ‘Landau classification’, in which phases of matter are distinguished by symmetry breaking
- instead, phases are distinguished by some ’topological quantum number’
  - classic example (2+1) is the Hall conductivity σ_H
  - associated with transport $\langle J_\mu(x) J_\nu(0) \rangle$
  - effective action is Chern-Simons for gauge field conjugate to charge current

Topological Insulators

- focus of this talk:
  - there are other transport properties (related to $\langle T_{\mu\nu}(x) T_{\lambda\rho}(0)\rangle$) that may be of interest
  - here the analogue of σ_H is the *dissipationless viscosity* ζ_H
- effective action is a functional of background 'gravitational' fields
- the dissipationless viscosity (2+1) is associated with a Chern-Simons-like term involving *torsion*
- natural to use the first-order formalism in which vielbein and spin connection are thought of as independent (but non-dynamical here)
- encounter interesting renormalization features
- many interesting questions remain to be answered
  - e.g., relation to anomalies

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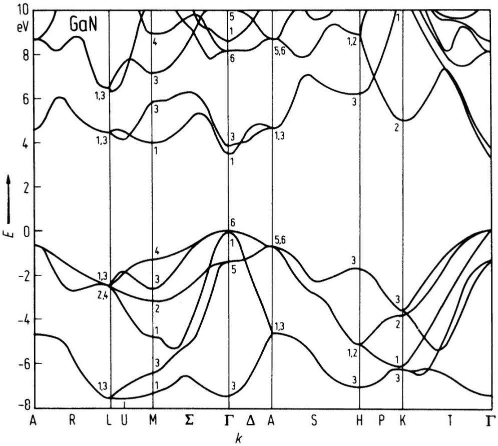
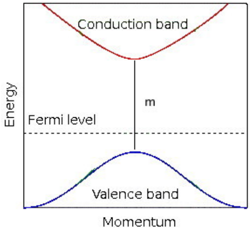


Topological Insulators

- topological features of the band structure of a material play a central role
- properties can be stated very generally (e.g., in terms of Chern classes of Berry curvatures, in which the parameter space is momentum space)
- often, simple models such as massive Dirac fields are a good approximation
- this can be thought of as zooming into some particular feature of the band structure
- in this case, the mass of the Dirac fermion is the parameter that sets the gap



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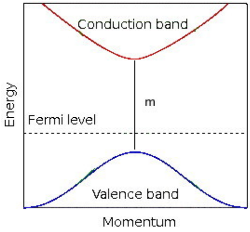

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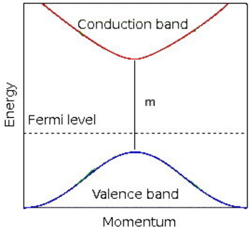
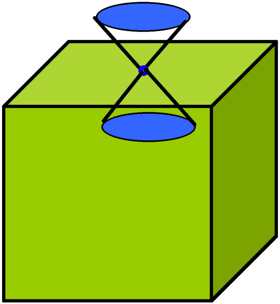


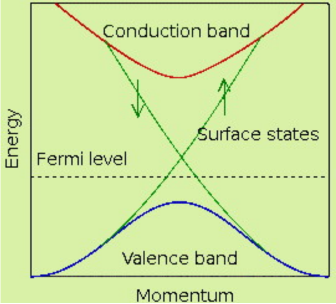
Topological Insulators: Axion Domain Walls

- as we will see, topological phases can be distinguished by the sign of the fermion mass
  - e.g., in 2+1, well-known that parity switches the sign of m
- axion domain wall (3+1): place materials of opposite sign of m next to one another
  - realistic example: HgTe vs. CdTe – very similar materials, but one has a very strong spin-orbit coupling that induces *band inversion* of s,p levels
  - i.e., if we imagine turning on couplings slowly, one material has a level crossing, the other doesn't
  - model gaps by Dirac fermions with opposite sign mass

Topological Insulators: Midgap Surface States

- on boundaries or on interfaces between phases, there are protected gapless fermionic states
  - an application of Callan-Harvey effect





Hall Conductivity: QHE

- the QHE is a time-reversal breaking topological insulator in which a magnetic field is applied perpendicular to a plane
- the low energy physics is described by a Chern-Simons action
  - couple a background EM field, effective action is CS in IR
  - the level of the EM CS determines the Hall conductivity

$$S_{\rm eff} = \frac{k}{4\pi}\frac{e^2}{\hbar}\int A\wedge dA \qquad \rightarrow \begin{cases} J^i = & \frac{ke^2}{h}\epsilon^{ij}E_j \\ J^0 = & \frac{ke^2}{h}B \end{cases}$$

- the Hall conductivity is quantized in units of e^2/h
- fundamental length scale $\ell_B^2 = \frac{\hbar c}{eB}$
- well-known that chiral CFT is supported on boundaries when $\sigma_H \neq 0$

QHE as Berry Curvature

- the Hall conductivity can be understood in terms of a Berry connection on momentum space, written as a sum over occupied states

$$a_i(\vec{k}) = -i \sum_{\alpha \in occ} \langle \alpha, \vec{k} | \frac{\partial}{\partial k_i} | \alpha, \vec{k} \rangle$$

- the Kubo formula then gives

$$\sigma_H = \frac{e^2}{h} \frac{1}{2\pi} \int_{BZ} f = \frac{e^2}{h} c_1$$

where

$$f_{ij}(\vec{k}) = \frac{\partial a_j(\vec{k})}{\partial k_i} - \frac{\partial a_i(\vec{k})}{\partial k_i j}$$

- extends to other topological insulators

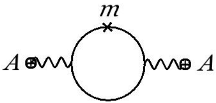
Quantum Anomalous Hall Effect (QAHE)

- other systems in 2+1 can also have a Hall conductivity [Haldane '88], depending on the symmetries of the system
- the simplest realization is just a massive Dirac fermion in 2+1
  - we'll use this model as our paradigm
  - it has the advantage that the band structure is simple, and calculations are straightforward
  - realistic systems, with more complicated band structure can be usefully modeled this way to some extent
- the Hall conductivity is sensitive to the *sign* of the fermion mass
- closely related to the parity anomaly

Hall Conductivity: QAHE

- in the Dirac model, this is obtained by computing the 1-loop determinant in a gauge field background

$$S = \int d^3x\, i\left[\frac{1}{2}\overline{\psi}\gamma^a D_a\psi - \frac{1}{2}\overline{D_a\psi}\gamma^a\psi - m\overline{\psi}\psi\right]$$

- 

$$\begin{aligned} S_{eff} &= \int \frac{d^3k}{(2\pi)^3} A_a(-k)\left[\int \frac{d^3p}{(2\pi)^3} tr\gamma^a(\not{p}-\not{k}-m)^{-1}\gamma^b(\not{p}-m)^{-1}\right]A_b(k) \\ &\simeq -m\left[\int \frac{d^3p}{(2\pi)^3}\frac{1}{(p^2-|m|^2)^2}\right]\int \frac{d^3k}{(2\pi)^3} A_a(-k) tr\gamma^a\not{k}\gamma^b A_b(k) \\ &\simeq -\frac{m}{|m|}\frac{1}{8\pi}\int \frac{d^3k}{(2\pi)^3}\epsilon^{abc}A_a(-k)k_c A_b(k) \end{aligned}$$

- this (apparently) finite result thus gives $\sigma_{xy} = \pm\frac{1}{2}\frac{e^2}{h}$
  - we'll return to a more careful regularization later

Elasticity and Viscosity

- consider an elastic medium with a fixed reference state, and consider small perturbations $u^a(x)$.

$$T^{ij} = \Lambda^{ijk\ell} u_{k\ell} + \eta^{ijk\ell} \dot{u}_{k\ell}$$

$$u_{k\ell} \simeq \frac{1}{2}\left(\frac{\partial u^k}{\partial x^\ell} + \frac{\partial u^\ell}{\partial x^k}\right) \quad (\textit{strain})$$

- we decompose

$$\Lambda^{ijk\ell} = \kappa \delta^{k\ell}\delta^{ij} + \mu\left(\delta^{ik}\delta^{j\ell} + \delta^{jk}\delta^{i\ell} - \frac{2}{D}\delta^{ij}\delta^{k\ell}\right)$$

where κ and μ are the bulk and shear moduli, respectively

$$\eta^{ijk\ell} = \zeta \delta^{k\ell}\delta^{ij} + \eta\left(\delta^{ik}\delta^{j\ell} + \delta^{jk}\delta^{i\ell} - \frac{2}{D}\delta^{ij}\delta^{k\ell}\right)$$

where ζ and η are the bulk and shear viscosities, respectively

Dissipationless Viscosity

- in $D = 2$ however, there is an additional possibility, since ϵ_{ij} is an invariant tensor
- there is an additional viscosity term proportional to

$$\eta_3^{ij;k\ell} = \frac{1}{4}\zeta_H \left(\epsilon^{ik}\delta^{j\ell} + \epsilon^{jk}\delta^{i\ell} + \epsilon^{i\ell}\delta^{jk} + \epsilon^{j\ell}\delta^{ik}\right)$$

- this is dissipationless, so the coefficient is called *dissipationless viscosity* (or *Hall viscosity*)
- note that this is entirely analogous to the current $J^i = \sigma^{ij} E_j$ and in $D = 2$ we can write

$$\sigma^{ij} = \sigma_S^{ij} + \sigma_H \epsilon^{ij}$$

and σ_H is the (dissipationless) Hall viscosity

The Geometry of Strain

- one should think in terms of a vielbein (co-frame)

$$
\begin{aligned}
e^a_\mu &= \delta^a_\mu + w^a_\mu \\
w^a_\mu &= \frac{\partial u^a}{\partial x^\mu}
\end{aligned}
$$

- we will see that it is e^a that is analogous to the gauge field A, and the torsion T^a plays the role of the curvature F
- torsion enters because we have a (non-trivial) material medium
- it is natural to ask:
  - is the dissipationless viscosity an interesting observable (analogous to σ_H)?
  - is it in some sense topological?

First-order Formalism

- recall that in differential geometry, if we consider the bundle of (co)frames $\{e^a\}$ on a manifold, we can endow it with a connection; given a section of this bundle, we have a metric
- the connection is referred to as *metric compatible* if it kills the metric, $\nabla g = 0$
  - this implies that the connection can be thought of as one-forms ω^{ab} valued in $SO(n)$ (rather than $SL(n)$)
- there is always a unique (Levi-Civita) connection, which is torsion-free

$$T^a \equiv de^a + \omega^a{}_b \wedge e^b$$

- of course, in (the metric formulation of) classical GR, the vanishing of the torsion is a *constraint*, which greatly complicates the theory

First-order Formalism

- in the first-order formalism, we regard e^a and $\omega^a{}_b$ as independent variables, and there are formulations of classical GR in which the torsion-free condition comes about as an equation of motion rather than a constraint ['Einstein-Cartan', 'MacDowell-Mansouri']
- in this talk, I'm interested in e^a and $\omega^a{}_b$ as background (non-dynamical) fields, much like EM fields are considered as background fields in CMT
- in this context, there is no reason whatsoever to assume that the torsion should vanish
- thinking this way has consequences...

Stress and Lorentz Currents

- consider the stress-energy tensor. One often claims that in field theory this can be written as $\frac{\delta S}{\delta g^{\mu\nu}}$
- what's really going on there is that if one takes a theory without dynamical gravity, one can introduce a metric and require that the resulting theory is diff- and Lorentz-invariant
- this has the effect of relating the canonical stress tensor to $\frac{\delta S}{\delta g^{\mu\nu}}$
- in the first order formalism, we can do the same thing
  - we find currents associated with both diffeomorphisms and Lorentz transformations

Stress and Lorentz Currents

- associated with Lorentz transformations

$$\begin{aligned} \delta e^a &= \theta^a{}_b e^b \\ \delta\omega^a{}_b &= ([\theta,\omega] - d\theta)^a{}_b = -D(\theta^a{}_b) \end{aligned}$$

  there is a Lorentz current $J_a^{\mu b} = \frac{\delta S}{\delta \omega_\mu{}^a{}_b}$

- associated with diffeomorphisms

$$\begin{aligned} \delta' e^a &= i_\xi T^a + D(i_\xi e^a) \\ \delta' \omega^a_b &= i_\xi R^a{}_b \end{aligned}$$

  there is a 'stress current' $J_a^\mu = \frac{\delta S}{\delta e_\mu^a}$

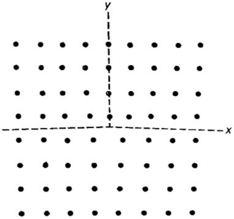
- if we go back to the metric formulation with LC connection, we recover $T_{\mu\nu}$

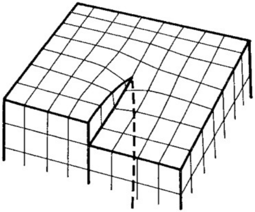
Sources of Torsion: Dislocations

- in general, a dislocation has the property that passage around any closed contour enclosing a dislocation line, results in a translation of the displacement vector *u* by the Burgers vector *b*

$$\oint du^a = -b^a$$

- several types of dislocation are possible





Sources of Torsion: Dislocations

- curvature is associated instead with disinclinations – a rotation (or Lorentz transformation) after following a closed path
- these statements fit well with classical geometry – indeed, one has, in the presence of torsion

$$[\nabla_{e_a}, \nabla_{e_a}] = -T^c_{ab}\nabla_{e_c} + R_{cd;ab}J^{cd}$$

The Effective Action

- so let's compute the dissipationless viscosity in a simple topological insulator
- in fact, there is a unique parity odd, Lorentz invariant with one derivative that can occur in $2+1$ dimensions

$$S_{eff} = \frac{\zeta_H}{4\pi} \int e^a \wedge T^b \eta_{ab}$$

so this is what we are looking for

- whereas in $3+1$, we can write $tr\ F \wedge F = d\ tr(A \wedge dA + \frac{2}{3}A^3)$, here we have

$$d(e^a \wedge T^b \eta_{ab}) = T^a \wedge T^b \eta_{ab} - e^a \wedge e^b \wedge R_{ab}$$

- the latter is the *Nieh-Yan form*: is it topological?

ζ_H for IQHE

- consider first IQHE: the computation can be done by putting the system on a torus, and varying the metric at constant volume

$$e^1 = \frac{1}{\sqrt{\tau_2}}(dx - \tau_1 dy), \qquad e^2 = \sqrt{\tau_2} dy$$

- the wavefunctions in the LLL can be written ($z = (x + \tau y)/L$)

$$\psi_0^{(s)}(x, y) \;=\; e^{i\pi N_\Phi \tau y^2/L^2} e^{i\pi(1+2s)z} \theta\left(N_\Phi z + s\tau; N_\Phi \tau\right)$$

for $s = 0, 1, ..., N_\Phi - 1$ (finite number of states)

- the system has a Berry curvature (in space of metrics), and this can be related to ζ_H

IQHE [Avron et al '95]

$$F = \frac{N_\Phi}{8\pi} \frac{d\tau_1 \wedge d\tau_2}{\tau_2^2} \quad \rightarrow \quad \zeta_H = \frac{\hbar}{8\pi \ell_B^2}$$

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ζ_H for QAHE (massive Dirac)

- we did the calculation in three different ways
  - as a Berry curvature on a torus
  - as a Feynman diagram calculation of a stress-stress correlator $\langle J_a^\mu(x) J_b^\nu(0)\rangle$
  - as an effective action: Dirac determinant in background e^a, $\omega^a{}_b$ fields
- I'll describe the latter here; the Dirac operator is

$$\not{D} = \gamma^a e_a^\mu \left(\partial_\mu + \frac{1}{4}\omega_{\mu;bc}\gamma^{bc}\right) + T^a_{ba}\gamma^b$$

- we may write

$$-\ln\det(\not{D} - m) = -\ln\det(\not{D}_{LC} - m) - \ln\det\frac{\not{D} - m}{\not{D}_{LC} - m}$$

- the LC determinant will contain the volume divergence, gravitational CS, ...
- the latter relative factor vanishes with torsion

ζ_H for QAHE (massive Dirac)

- $$\begin{aligned} -\ln\det\frac{\not{D}-m}{\not{D}_{LC}-m} &= -\ln\det\frac{\not{D}-m}{\not{D}_{LC}-m}\frac{\not{D}_{LC}+m}{\not{D}_{LC}+m} \\ &= -\ln\det\left(1+\frac{(\not{D}-\not{D}_{LC})(\not{D}_{LC}+m)}{\not{D}_{LC}^2-m^2}\right) \end{aligned}$$
- after a short computation, we find $\not{D}-\not{D}_{LC}=\pm\frac{1}{4}\varepsilon^{abc}C_{a;bc}\equiv\frac{1}{4}c$ (here, $C_{a;bc}$ is the *contorsion*)
- thus there is a simple perturbative expansion in powers of c that we can do

$$\begin{aligned} -\ln\det\frac{\not{D}-m}{\not{D}_{LC}-m} &= -\ln\det\left(1+\frac{1}{4}c(\not{D}_{LC}-m)^{-1}\right) \\ &\simeq -\frac{1}{4}tr\ c(\not{D}_{LC}-m)^{-1}+\ldots \end{aligned}$$

ζ_H for massive Dirac

- working close to flat space, we then easily get

$$\begin{aligned} & \frac{1}{2} m \int d^3x\, c(x) \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+m^2} \\ = \; & \frac{1}{2} imI_T(m) \int d^3x\, c(x) = \pm\frac{1}{2} imI_T(m) \int e^a \wedge T_a \end{aligned}$$

- the integral $I_T(m)$ that appears here is linearly divergent

$$I_T(m) = -\int_\epsilon^\infty dt \int \frac{dk}{2\pi^2} e^{-t(k^2+m^2)} k^2 = -\frac{1}{4\pi}\left(\frac{1}{\sqrt{\pi\epsilon}} - |m| + \ldots \right)$$

ζ_H for massive Dirac

- the presence of this divergence is easy to understand by power-counting: e^a_μ has mass dimension zero (c.f. A_μ, mass dimension one)
  - thus, the coefficient of $e^a \wedge T_a$ must be of order $m\Lambda$
  - the coefficient of $A \wedge dA$ must be of order $m/|m|$ (or smaller)
- in fact both are sensitive to the cutoff
- this can be seen in a lattice version

$$H = \sum_{p_x, p_y} c^\dagger_{\vec{p}} \left[\sin p_x \sigma_x + \sin p_y \sigma_y + (2 - m - \cos p_x - \cos p_y)\sigma_z\right] c_{\vec{p}}$$

  - here there are four Dirac points at $\vec{p} = (k\pi/a, \ell\pi/a)$ $(k, \ell = 0, 1)$

$$H_{(k,\ell)} \simeq e^{ik\pi} p_x \sigma_x + e^{i\ell\pi} p_y \sigma_y + (2k + 2\ell - m)\sigma_z = \sum_a d_a(k, \ell; m)\sigma_a$$

  - each contributes to both σ_H and ζ_H

ζ_H for massive Dirac: regulated version

- each Dirac point contributes $\pm\frac{1}{2}$ (depending on sign of mass) to $\sigma_H \to \sigma_H \in \mathbb{Z}$
- whereas σ_H is about *charge*, ζ_H is about *momentum*
- thus each Dirac point contributes to ζ_H, but three of them contribute $O(1/a)$
- shows up in the continuum as a linear divergence
- in the continuum theory, we can regulate via Pauli-Villars

$$\sigma_H^{reg} = \frac{e^2}{2h}\sum_{i=0}^{N} C_i \; sign \; M_i, \qquad \zeta_H^{reg} = \frac{1}{16\pi}\sum_{i=0}^{N} C_i M_i I_T(M_i)$$

(here $C_0 = 1, M_0 = m$)

ζ_H for massive Dirac: renormalization

- there is a physical renormalization condition that we apply:
  - if $\sigma_H = 0$, then $\zeta_H = 0$
  - here, the idea is that if $\sigma_H = 0$ the system should be trivial (no d.o.f. induced on boundaries)
  - secondly, require ζ_H to be finite $\rightarrow \sum_{i=0}^{N} C_i = 0$
- suppose that $m < 0$ is the trivial insulator. We conclude

$$-1 + \sum_{i=1}^{N} C_i \; sign \; M_i = 0, \qquad -|m|^2 + \sum_{i=1}^{N} C_i |M_i|^2 \; sign \; M_i = 0$$

- then, for $m > 0$, we get

$$\sigma_H^{reg} = \frac{e^2}{h}, \qquad \zeta_H^{reg} = \frac{1}{16\pi} \frac{|m|^2}{2\pi} \equiv \frac{\hbar}{8\pi\ell^2}$$

- the viscosity $\rightarrow 0$ as $m \rightarrow 0$, much as it does as $B \rightarrow 0$ in IQHE

Conclusions and Comments

- related phenomena exist in other dimensions
- one puzzle here is the relation to anomalies
  - there are various physical processes in the EM case that one can consider that can be related to spectral flow
  - there are analogues for torsion/dislocations, but the results cannot be interpreted in terms of zero modes alone (again, because torsion couples to momentum rather than charge)