Real-Time, Finite-Temperature AdS/CFT

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Real-Time, Finite-Temperature AdS/CFT

Work done with

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? Question ?

How to calculate n-point correlators like

$$\langle \mathcal{T} O(x_1) O(x_2) O(x_3) \rangle$$

in a ***Strongly Coupled***, ***Finite Temperature*** field theory?

- Strong Coupling: AdS/CFT duality
- Finite Temperature: AdS-Schwarzschild/Thermal CFT duality

QCD (Quark-Gluon Plasma) *when g is large* *but slow-changing*	$\Longrightarrow$	$\mathcal{N} = 4$, $SU(N_c)$ Supersymmetric Yang-Mills (SYM) Field Theory

Outline

- A Brief Review
  - CFT: $\mathcal{N} = 4 \; SU(N_c)$ Super-YM Field Theory
  - AdS: Type IIB Supergravity in AdS_5 Space
  - $T = 0$ Duality: AdS/CFT
  - $T \neq 0$ Duality: AdS-Schwarzschild/Thermal CFT
- Scalar in AdS-Schwarzchild
  - Bulk-to-Boundary Propagators
  - 2-Point Functions
  - 3-Point Functions
  - related to Schwinger-Keldysh Formalism
- Summary & Outlook

$\mathcal{N} = 4\ SU(N_c)$ Super-YM Field Theory

$$\mathcal{L} = tr\Big\{ -\sum_{i=1}^{6} D_\mu \phi^i D^\mu \phi^i - \sum_{I=1}^{4} i\bar{\psi}^I \gamma^\mu D_\mu \psi^I - \frac{1}{2g_{YM}^2} F_{\mu\nu} F^{\mu\nu} + \cdots \Big\}$$

- $d = 4$ flat Minkowskian space
  Poincaré invariance (rotation and boost + translation)
  Symmetry group $\Rightarrow SO(1,3)$
- Maximal supersymmetry
  - Bosons: $\phi^i(x)$ $(i = 1..6)$, $A_\mu(x)$
  - Fermions: $\psi^I_\alpha(x)$ $(I = 1..4)$

  R-symmetry: mix same spin fields
  Symmetry group $\Rightarrow SO(6)$
- $\beta(g_{YM}) = 0$ at quantum level
  Scale invariance $\Rightarrow$ ***Conformal*** field theory
  Symmetry group $\Rightarrow SO(2,4)$ (together with Poincaré)
- Global symmetry: $\mathbf{SO(2,4) \otimes SO(6)}$

$AdS_5 \times S^5$ **Geometry**

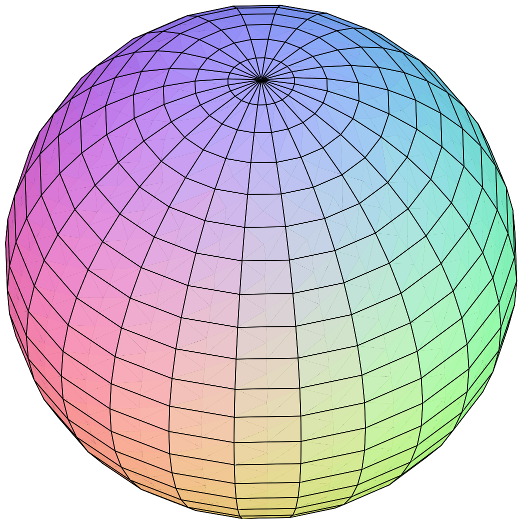
- $SO(6) \quad \Longrightarrow \quad \mathbf{S^5}$ (5-d sphere)

  Embedding:

  $$X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = R^2$$

  Metric:

  $$ds^2 = R^2 d\Omega_5^2$$



- $SO(2,4) \quad \Longrightarrow$ **?**

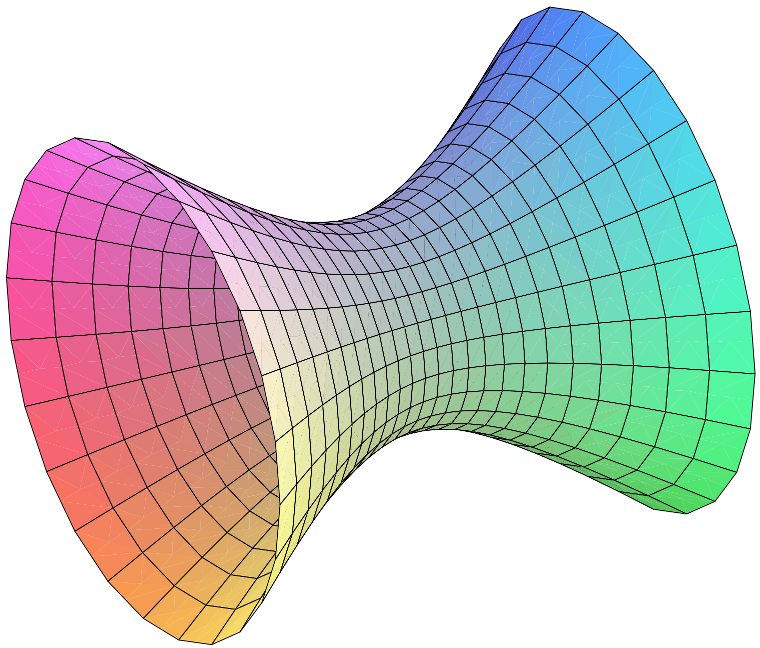
$AdS_5 \times S^5$ Geometry

- $SO(2,4) \quad \Longrightarrow \quad$ **$\mathbf{AdS_5}$** (5-d Anti-de Sitter space)

  Embedding:
  $$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -R^2$$

  Metric:
  $$\begin{aligned} ds^2 &= \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 \\ &= \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) \end{aligned}$$

- Curvature: negative constant
- 2 Poincaré patches
- 1 boundary in each Poincaré patch: at $r = \infty$ or $z = 0$



Type IIB Superstring Theory

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi \sqrt{\gamma} \Big\{ [\gamma^{mn} G_{\mu\nu}(X) + \epsilon^{mn} B_{\mu\nu}(X)] \partial_m X^\mu(\xi) \partial_n X^\nu(\xi) + \alpha' \mathcal{R}^{(2)}_\gamma \Phi(X) \Big\}$$
$$+ \frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi \sqrt{\gamma} G_{\mu\nu}(X) \Big[\psi^\mu_+(\xi) D_{\bar{z}} \psi^\nu_+(\xi) + \psi^\mu_-(\xi) D_z \psi^\nu_-(\xi) \Big] + \cdots$$

- String $\Rightarrow$ 1 spatial dimension $\Rightarrow$
  $d = 2$ worldsheet: $\xi^m = (\tau, \sigma)$, $(m = 0, 1)$
  Worldsheet metric: $\gamma_{mn}(\xi)$
- Bosonic field: embedding space
  $d = 10$ spacetime: $X^\mu = (t, \vec{x}, r, ...)$, $(\mu = 0..9)$
  Target space metric: $G_{\mu\nu}(X)$
- Fermionic field: $\psi^\mu_\pm(\xi)$
- ***Coupling constant***: g_s, ***string length***: $\alpha' = l_s^2$
- $AdS_5 \times S^5$ is the backgound

Type IIB Supergravity (SUGRA): $d = 10$

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}X \sqrt{-G} e^{\Phi} \left\{ \mathcal{R}^{(10)} + 4\partial_\mu \Phi(X) \partial^\mu \Phi(X) \right\} + \cdots$$

where $\kappa_{10} = 8\pi^{\frac{7}{2}} g_s l_s^4$

- ***Low Energy (Point) limit*** of string: massive string modes decouple, only massless mode survives.

$$\frac{l_s}{R} \to 0$$

- $d = 10$ spacetime: $X^\mu = (t, \vec{x}, r, ...)$, $(\mu = 0..9)$
- Fields: metric $G_{\mu\nu}(X)$, scalar $\Phi(X)$, ...
- $G_{\mu\nu}(X) = AdS_5 \times S^5$ is a classical solution
- ***Classical limit***: suppress quantum fluctuations

$$g_s \to 0$$

Classical Type IIB Supergravity: $d = 5$

- Kaluza-Klein reduction from $d = 10$ to $d = 5$, eg:

$$\Phi(X) = \phi(t, \vec{x}, r) Y_{lm\ldots}(\Omega_5)$$

  integrate over S^5

- Eigenvalue of $Y_{lm\ldots}$ looks like a mass term in $d = 5$

$$\Box_{S^5} = \Delta(\Delta - 4) \Rightarrow m^2$$

- 5-d effective action

$$S = \frac{N^2}{8\pi^2 R^3} \int d^5x \sqrt{-g} \Big\{ \mathcal{R}^{(5)} - \frac{1}{2} \left(\partial^\mu \phi \partial_\mu \phi + m^2 \phi^2 \right) - \frac{R^2}{8} F^a_{\mu\nu} F^{a\mu\nu} + \cdots \Big\}$$

  This is the ***starting point*** of an actual AdS calculation.

String/Gauge Duality

2 quite different theories:

String/SUGRA vs. CFT

Same global symmetry groups

$\longrightarrow$ Prerequisite of forming a duality

Duality: Mapping of everything

- Map of (coupling) constants
- Map of String fields $\Longleftrightarrow$ CFT operators
- Map of String action $\Longleftrightarrow$ CFT generating functional
- $\longrightarrow$ Map of correlation functions

AdS/CFT Duality: (Coupling) Constants

Quantum IIB String in $AdS_5 \times S^5$ (g_s, l_s)		$\mathcal{N}=4 \quad SU(N_c)$ SYM (CFT) $(N_c, \lambda = g^2_{YM} N_c)$
$g_s \to 0 \quad \downarrow \quad \frac{l_s}{R} \to 0$	$4\pi g_s = \frac{\lambda}{N_c}$ $\left(\frac{R}{l_s}\right)^4 = \lambda$	$\lambda \to \infty \quad \downarrow \quad N_c\big|_\lambda \to \infty$
Weakly coupled IIB SUGRA in AdS_5	$\Longleftrightarrow$	***Strongly coupled*** large N_c Quantum CFT

AdS/CFT Duality: Fields / Operators

AdS Fields		CFT Operators
ϕ	$\longleftrightarrow$	$\mathcal{L}$
A^a_μ	$\longleftrightarrow$	J^a_μ
$g_{\mu\nu}$	$\longleftrightarrow$	$T_{\mu\nu}$
	$\vdots$	
m	$\longleftrightarrow$	Δ

AdS/CFT Duality: Action & Correlators

AdS (classical)		CFT (strongly coupled)
$\lim_{z\to 0} \frac{\phi(x,z)}{z^{2-\frac{\Delta}{2}}} = \bar{\phi}(x)$	$\bar{\phi}(x) = J(x)$ $\longleftrightarrow$	$J(x) \Rightarrow O(x)$
$S_{cl}[\bar{\phi}] = \int d^5x \sqrt{-g}\,(\partial\phi)^2 + \cdots$	$e^{iS_{cl}[\bar{\phi}]} = Z[J]$ $\longleftrightarrow$	$Z[J] = \int \mathcal{D}\phi e^{iS + i\int d^4x J(x) O(x)}$

$$\left.\frac{\delta^n S_{cl}[\bar{\phi}]}{\delta\bar{\phi}(x_1)\delta\bar{\phi}(x_2)\ldots}\right|_{\bar{\phi}\to 0} \Rightarrow \langle O(x_1)O(x_2)\ldots\rangle \sim \left.\frac{\delta^n \ln Z[J]}{\delta J(x_1)\delta J(x_2)\ldots}\right|_{J\to 0}$$

AdS-Schwarzschild/CFT Duality: $T \neq 0$

Weakly coupled IIB SUGRA in $AdS-$***Schwarzschild***$_5$ $\mathbf{T_H}$	$\Longleftrightarrow$ *[Witten]*	Strongly coupled Quantum CFT at ***Finite Temperature*** $\mathbf{T}$

- AdS_5:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2$$

- $AdS-Schwarzschild_5$:

$$ds^2 = \frac{r^2}{R^2}(-f dt^2 + d\vec{x}^2) + \frac{R^2}{r^2 f}dr^2$$

$$\begin{cases} f = 1 - \left(\frac{r_0}{r}\right)^4 & \text{- Blackening function} \\ T_H = \frac{r_0}{\pi R^2} & \text{- Hawking temperature} \end{cases}$$

AdS-Sch/CFT Duality: Procedure & Complications

1. Solve EOM in AdS-Schwarzschild background

$$(\Box - m^2)\phi(t,\vec{x},r) = 0 \qquad \text{with} \qquad \Box = \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu$$

- Find 2 eigenfunctions
  - AdS: 2 poles at $r = 0$ and $r = \infty$ $\Longrightarrow$ Bessel eq.
  - AdS-Sch: ***4 poles***, 2 extra from f at $r^2 = \pm r_0^2$ $\Longrightarrow$ **?**
- Superposition by ***B.C.***'s $\Longrightarrow$ **?**

2. Calculate classical on-shell action

$$S_{cl}[\bar{\phi}] = \int d^5x \sqrt{-g}\left\{\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2\right\}$$

global structure: ***4 quadrants*** + 2 timelike boundaries, so $\int d^5x =$ **?**

3. Take functional derivative of $S_{cl}[\bar{\phi}]$

ODE with 4 Regular Poles: Heun's Equation

$$\frac{d^2\phi(z)}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)\frac{d\phi(z)}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}\phi(z) = 0$$

- 4 poles: $z = 0,\ 1,\ a,\ \infty$
- Regularity at $z = \infty$: $\quad \epsilon = \alpha + \beta - \gamma - \delta + 1$
- Solution: Heun's function in $|z| \in [0, \min\{1, |a|\})$

$$Hl(a, q, \alpha, \beta, \gamma, \delta\,; z) = \sum_{k=0}^{\infty} c_k z^k = 1 + \frac{q}{\gamma a} z + \cdots$$

- Recursion relation: $\quad c_{k+1} = \ldots c_k + \ldots c_{k-1}$
- 2 independent solutions (eigenfunctions):

$$\begin{aligned}
\phi_1(z) &= Hl(a, q, \alpha, \beta, \gamma, \delta\,; z) \\
\phi_2(z) &= z^{1-\gamma} Hl(a, q - (\gamma-1)(a\delta+\epsilon), \beta-\gamma+1, \alpha-\gamma+1, 2-\gamma, \delta\,; z)
\end{aligned}$$

or $\quad \phi_3(z) = \lim_{\gamma\to -1}\left\{\phi_1(z) - \frac{c_2(\gamma,\cdots)}{\gamma+1}\phi_2(z)\right\}$

Solving Scalar EOM in AdS-Schwarzschild

$$(\Box - m^2)\,\phi(t,\vec{x},r) = 0 \qquad \text{with} \qquad \Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$$

$$\Downarrow$$

$$\begin{cases} \phi(t,\vec{x},r) \xrightarrow[\omega=\frac{E}{2\pi T},\quad \vec{k}=\frac{\vec{p}}{2\pi T}]{\mathcal{F}\quad \text{w.r.t.}\quad (t,\vec{x})} \phi(\omega,\vec{k}\,;u) \\ u = \left(\frac{r_0}{r}\right)^2 \quad \longrightarrow \quad f = 1-u^2 \begin{cases} \text{boundary:} & u=0 \\ \text{horizon:} & u=1 \end{cases} \end{cases}$$

$$\Downarrow$$

$$\left\{\frac{\partial^2}{\partial u^2} + \frac{u^2+1}{u(u^2-1)}\frac{\partial}{\partial u} + \frac{\omega^2}{u(u^2-1)^2} + \frac{\vec{k}^2}{u(u^2-1)} + \frac{m^2R^2}{4u^2(u^2-1)}\right\}\phi(\omega,\vec{k}\,;u) = 0$$

$$\Downarrow$$

$$\begin{cases} \phi(\omega,\vec{k}\,;u) = F(\omega,\vec{k}\,;u)\,\bar{\phi}(\omega,\vec{k}) \\ \left.\frac{F(\omega,\vec{k}\,;u)}{u^{2-\frac{\Delta}{2}}}\right|_{u\to 0} = 1 \end{cases} \rightarrow \begin{cases} F(u)\text{: } \textbf{\textit{bulk-to-boundary propagator}} \\ \bar{\phi}(\omega,\vec{k})\text{: boundary field} \end{cases}$$

AdS-Schwarzschild: Bulk-to-Boundary Propagators (1)

i.e., 2 independent solutions of the EOM: *[Starinets]*

$$F(\omega,\vec{k}\,;u) = N_{\omega,\vec{k}}\, u^{2-\frac{\Delta}{2}}\,(1+u)^{\frac{\omega}{2}}\,(1-u)^{-i\frac{\omega}{2}}\, Hl\,(a,q,\alpha,\beta,\gamma,\delta\,;1-u)$$

with
$$\begin{cases} \alpha = \beta = \left(2-\frac{\Delta}{2}\right) + \frac{1-i}{2}\,\omega \\ \gamma = 1 - i\omega \\ \delta = 3 - \Delta \\ a = 2 \\ q = \left(4 - 2\Delta + \frac{\Delta^2}{4}\right) + \left[\frac{1}{2} + \left(\Delta - \frac{7}{2}\right) i\right]\omega - \frac{i}{2}\omega^2 - \omega^2 + \vec{k}^2 \end{cases}$$

The other is $F^*(\omega,\vec{k}\,;u)$.

What is $F(\omega, \vec{k}\,; u)$?

- $F(\omega, \vec{k}\,; u) \to F(\omega + i\epsilon, \vec{k}\,; u)$: analytic in UHP of ω
  $\Longrightarrow$ ***Retarded propagator***

- Near-horizon $u \to 1$ behavior

$$F(\omega, \vec{k}\,; u) \to (1-u)^{-i\frac{\omega}{2}}$$

inside the F.T.

$$e^{-iEt} F(\omega, \vec{k}\,; u) \to e^{-iE(t + \frac{R^2}{4r_0} \ln \frac{r-r_0}{r_0})} \sim V^{-i\omega}$$

$\Longrightarrow$ $F(\omega, \vec{k}\,; u)$ is $\begin{cases} \textbf{\textit{incoming wave in R}} \\ \text{outgoing wave in L} \end{cases}$

- $F(\omega, \vec{k}\,; u)$ has supports only ***inside the future light-cone***
  $\Longrightarrow$ Retarded propagator

Global Structure of AdS-Schwarzschild

Poincaré (local)

$$ds^2 = -\frac{r^2 f}{R^2} dt^2 + \frac{R^2}{r^2 f} dr^2$$

Coordinate transformation:

$$\begin{cases} t_K = \beta e^{\frac{\alpha R^2}{2r_0} r_*} \sinh \alpha t \\ r_K = \beta e^{\frac{\alpha R^2}{2r_0} r_*} \cosh \alpha t \end{cases} \quad \text{in R}$$

$$\begin{cases} t_K = -\beta e^{\frac{\alpha R^2}{2r_0} r_*} \sinh \alpha t \\ r_K = -\beta e^{\frac{\alpha R^2}{2r_0} r_*} \cosh \alpha t \end{cases} \quad \text{in L}$$

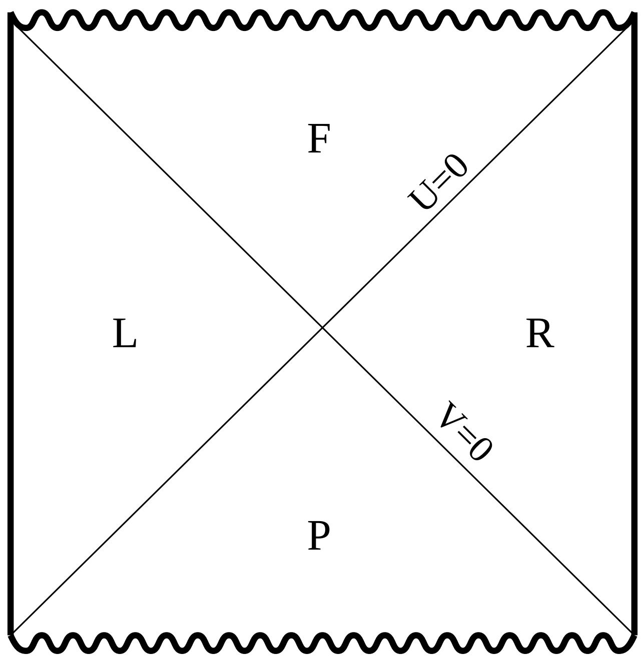
$$r_* = \arctan\left(\frac{r}{r_0}\right) - \operatorname{arctanh}\left(\frac{r_0}{r}\right)$$

$$\begin{cases} t_K = V + U \\ r_K = V - U \end{cases}$$

Kruskal (global)

$$ds^2 = e^{\Lambda(t_K, r_K)}(-dt_K^2 + dr_K^2)$$

AdS-Sch Penrose diagram



Prescriptions & Boundary Conditions

Defined in Kruskal (global) coordinates

- Action integral: only in ***L*** & ***R*** quadrants, '***minus***' *[Frolov & Martinez]*

$$\int \equiv \int_R - \int_L$$

- ***Continuity*** across the horizon: *[Unruh]*

$$f(U, V = 0) \quad f(U = 0, V)$$

- Frequency selection: *[Son & Herzog]*

Physical particles

$$\begin{cases} \textbf{\textit{R:}} & \textbf{\textit{incoming}} \\ \textbf{\textit{L:}} & \textbf{\textit{outgoing}} \end{cases}$$

AdS-Schwarzschild: Bulk-to-Boundary Propagators (2)

Bulk field (globally defined):

$$\phi(t_K, \vec{x}_K, r_k) = \begin{cases} \phi_R(t, \vec{x}, r) & \text{in R} \\ \phi_L(t, \vec{x}, r) & \text{in L} \end{cases}$$

$$\phi_R(t, \vec{x}, r) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} \left\{ f_{RR}(\omega, \vec{k}\,; u)\bar{\phi}_R(\omega, \vec{k}) + f_{RL}(\omega, \vec{k}\,; u)\bar{\phi}_L(\omega, \vec{k}) \right\}$$

$$\phi_L(t, \vec{x}, r) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} \left\{ f_{LR}(\omega, \vec{k}\,; u)\bar{\phi}_R(\omega, \vec{k}) + f_{LL}(\omega, \vec{k}\,; u)\bar{\phi}_L(\omega, \vec{k}) \right\}$$

Bulk-to-boundary propagators: ***Feynman*** & ***Wightman*** *[Son & Herzog]*

$$\begin{cases} f_{RR}(\omega, \vec{k}\,; u) = \frac{e^{2\pi\omega}}{e^{2\pi\omega}-1} F(\omega, \vec{k}\,; u) - \frac{1}{e^{2\pi\omega}-1} F^*(\omega, \vec{k}\,; u) \\ f_{RL}(\omega, \vec{k}\,; u) = \frac{e^{\pi\omega}}{e^{2\pi\omega}-1} \left[-F(\omega, \vec{k}\,; u) + F^*(\omega, \vec{k}\,; u) \right] \\ f_{LR}(\omega, \vec{k}\,; u) = \frac{e^{\pi\omega}}{e^{2\pi\omega}-1} \left[F(\omega, \vec{k}\,; u) - F^*(\omega, \vec{k}\,; u) \right] \\ f_{LL}(\omega, \vec{k}\,; u) = -\frac{1}{e^{2\pi\omega}-1} F(\omega, \vec{k}\,; u) + \frac{e^{2\pi\omega}}{e^{2\pi\omega}-1} F^*(\omega, \vec{k}\,; u) \end{cases}$$

where $e^{-2\pi\omega} = e^{-E/T}$ - Boltzman factor

Structure of On-Shell Action

$$S[\phi] = \int_R d^5x\sqrt{-g}\mathcal{L}[\phi] - \int_L d^5x\sqrt{-g}\mathcal{L}[\phi]$$

$$\mathcal{L}_0 = \partial_\mu\phi\partial^\mu\phi + m^2\phi^2, \qquad \mathcal{L}_{pert} = \frac{\lambda}{3!}\phi^3$$

Substitute 0^{th} order EOM solution into the action

$$S_{cl}[\phi] = \int_{R-L} d^4x\sqrt{-g}g^{uu}\phi\,\partial^u\phi\Big|_{u\to 0} + \frac{\lambda}{3!}\int_{R-L} dud^4x\sqrt{-g}\phi^3$$

$$S_{cl}[\bar{\phi}] = \int d^4p_1 d^4p_2 \delta^{(4)}(p_1+p_2)\sum_{ij} f_{ij_1}(p_1;u)\sqrt{-g}g^{uu}\partial^u f_{ij_2}(p_2;u)\Big|_{u\to 0}\bar{\phi}_{j_1}(p_1)\bar{\phi}_{j_2}(p_2)$$

$$+\frac{\lambda}{3!}\int_0^1 du\sqrt{-g}\int d^4p_1 d^4p_2 d^4p_3\delta^{(4)}(p_1+p_2+p_3)\sum_{ij} f_{ij_1}(p_1;u)f_{ij_2}(p_2;u)$$

$$f_{ij_3}(p_3;u)\bar{\phi}_{j_1}(p_1)\bar{\phi}_{j_2}(p_2)\bar{\phi}_{j_3}(p_3)$$

where $i,j = R,L$

Next: to take functional derivative w.r.t. $\bar{\phi}$.

AdS-Schwarzschid: 2-Point Functions

Define:

$$G_{ij}(p_1, p_2) = -(-)^{i+j} \frac{\delta^2 S_{cl}[\bar\phi]}{\delta\bar\phi_i(p_1)\delta\bar\phi_j(p_2)}\bigg|_{\bar\phi\to 0}$$

where $i, j = 1, 2 \quad (1 \equiv R,\ 2 \equiv L)$

2-point functions: $F(u) \equiv F(p; u) = F(\omega, \vec{k}\,; u)$ *[Son & Herzog]*

$$\begin{cases} G_{11}(p) = \frac{N^2}{8\pi^2 R^3}\sqrt{-g}g^{uu}\left[\frac{e^{2\pi\omega}}{e^{2\pi\omega}-1}F(u)\partial_u F(u) - \frac{1}{e^{2\pi\omega}-1}F^*(u)\partial_u F^*(u)\right]\bigg|_{u\to 0} \\ G_{12}(p) = -\frac{N^2}{8\pi^2 R^3}\sqrt{-g}g^{uu}\frac{e^{\pi\omega}}{e^{2\pi\omega}-1}\left[-F(u)\partial_u F(u) + F^*(u)\partial_u F^*(u)\right]\bigg|_{u\to 0} \\ G_{21}(p) = G_{12}(p) \\ G_{22}(p) = \frac{N^2}{8\pi^2 R^3}\sqrt{-g}g^{uu}\left[\frac{1}{e^{2\pi\omega}-1}F(u)\partial_u F(u) - \frac{e^{2\pi\omega}}{e^{2\pi\omega}-1}F^*(u)\partial_u F^*(u)\right]\bigg|_{u\to 0} \end{cases}$$

Same structure as the propagators: Retarded vs. Feynman & Wightman ?

AdS-Schwarzschid: 3-Point Functions

Define:

$$G_{ijk}(p_1,p_2,p_3) = (-)^{i+j+k}\frac{\delta^3 S_{cl}[\bar\phi]}{\delta\bar\phi_i(p_1)\delta\bar\phi_j(p_2)\delta\bar\phi_k(p_3)}\bigg|_{\bar\phi\to 0}$$

where $i,j,k = 1,2; \quad (1\equiv R,\ 2\equiv L)$

3-point functions:

$$G_{ijk}(p_1,p_2,p_3) = (-)^{i+j+k}\frac{N^2}{16\pi^2R^3}\lambda\,\delta^{(4)}(p_1+p_2-p_3)\int_0^1 du\sqrt{-g}$$
$$\Big[f_{Ri}(p_1;u)f_{Rj}(p_2;u)f_{Rk}(p_3;u) - f_{Li}(p_1;u)f_{Lj}(p_2;u)f_{Lk}(p_3;u)\Big]$$
$$= (-)^{i+j+k}\frac{N^2}{16\pi^2R^3}\lambda\,\delta^{(4)}(p_1+p_2-p_3)\int_0^1 du\sqrt{-g}$$
$$\Big[n\big(e^{2\pi\omega_1},e^{2\pi\omega_2},e^{2\pi\omega_3}\big)F(p_1;u)F(p_2;u)F(p_3;u)+\cdots\Big]$$

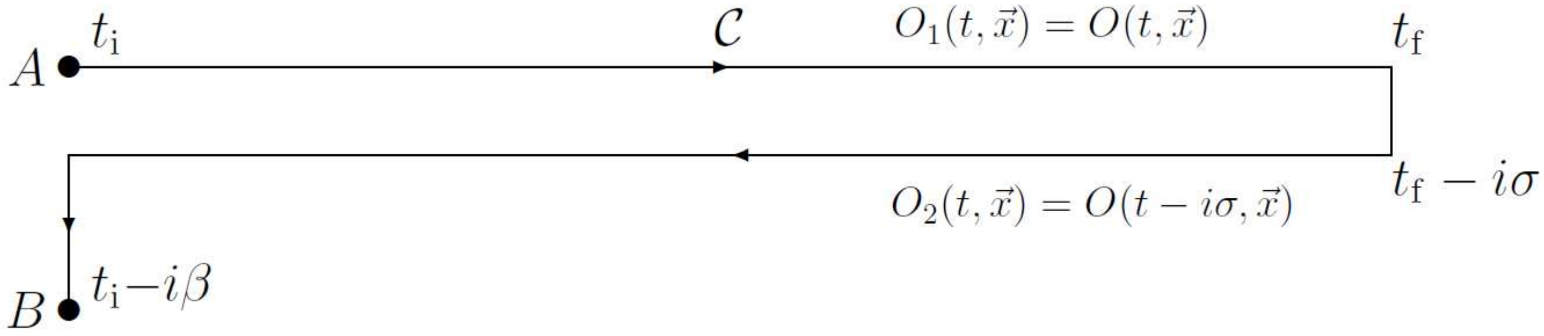
- ***Tree-level diagrams*** & ***Circling rules***

Schwinger-Keldysh Formalism for Field Theory at $T \neq 0$

- Real-time formalism
- Complex t-contour $\mathcal{C}$, ***doubler*** fields

$$S = \int_{\mathcal{C}} dt L$$

$$\boxed{e^{-\beta \hat{H}} \sim e^{it\hat{H}}}$$



- Generating function

$$Z[J_1, J_2] = \int \mathcal{D}\phi \, e^{iS + i\int d^4x J_1 O_1 - i \int d^4x J_2 O_2}$$

- n-point functions

$$G^{(n)}_{ab\ldots}(\{x_i\}) = -(-i)^{n-1} \frac{\delta^n \ln Z[J_1, J_2]}{\delta J_a(x_1) \delta J_b(x_2) \ldots} \qquad (a, b \ldots = 1, 2)$$

Mapping AdS-Sch Structure to Schwinger-Keldysh

- L & R structure $\Longleftrightarrow$ S-K contour
- 2 boundaries $\Longleftrightarrow$ 2 real time paths

$$R = 1, \quad L = 2$$

- Fields: right $\Leftrightarrow$ real, left $\Leftrightarrow$ doubler

$$\bar{\phi}_R = J_1, \quad \bar{\phi}_L = J_2$$

$$\phi_R = O_1, \quad \phi_L = O_2$$

- n-point function $n \times n \times \cdots$ matrices match, e.g.:

$$G_{RLRR\ldots} = G_{1211\ldots}$$

- Hawking temperature $\Longleftrightarrow$ imaginary contour period

$$\sigma = \frac{\beta}{2}$$

2-Point Function: Matrix & Causal

Schwinger-Keldysh 2-point function marix has the following properties:

- KMS (Kubo-Martin-Schwinger) relation ✓

$$G^*_{11}(p) = -G_{22}(p), \; G^*_{12}(p) = -G_{21}(p)$$

- Largest time identity $\quad (\pi\omega = \frac{E}{2T})$ ✓

$$G_{11} - e^{\pi\omega} G_{12} - e^{-\pi\omega} G_{21} + G_{22} = 0$$

- Feynman, Wightman & Causal ✓

$$G_{11}(p) = G_F(p), \quad G_{12}(p) = e^{\pi\omega} G^-(p), \quad G_R(p) = G_{11} - e^{\pi\omega} G_{12}$$

$$\begin{cases} G_{11}(p) = \Re G_R(p) + i \coth(\pi\omega) \Im G_R(p) \\ G_{12}(p) = i \,\mathrm{csch}(\pi\omega) \Im G_R(p) \end{cases}$$

$$\Downarrow$$

$$G_R(p) = \frac{N^2}{8\pi^2 R^3} \sqrt{-g} g^{uu} F(\omega, \vec{k}\,; u) \partial_u F(\omega, \vec{k}\,; u) \Big|_{u \to 0}$$

3-Point Function: Matrix & Causal

Schwinger-Keldysh 3-point function marix has the following properties:

- KMS relation ✓

$$G^*_{111}(p) = G_{222}(p),\ G^*_{121}(p) = G_{212}(p)\ \cdots$$

- Largest time identity $(p_3 = p_1 + p_2)$ ✓

$$G_{111} - e^{-\pi\omega_1} G_{211} - e^{-\pi\omega_2} G_{121} + e^{-\pi\omega_3} G_{221}$$
$$+ e^{\pi\omega_1} G_{122} + e^{\pi\omega_2} G_{212} - e^{\pi\omega_3} G_{112} - G_{222} = 0$$

- Feynman & Wightman

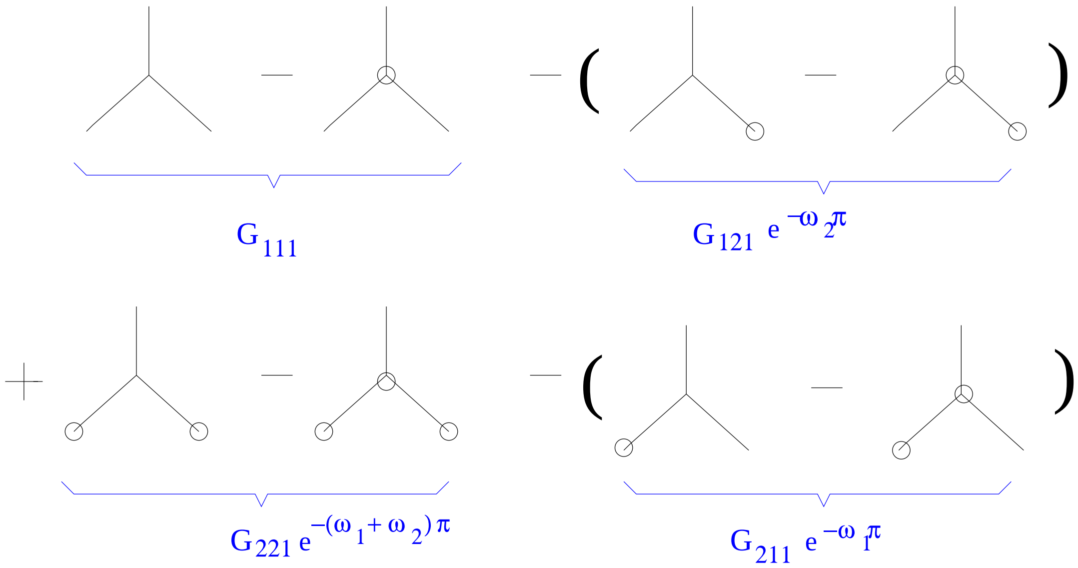
$$G_{111}(\{x\}) = G_F(\{x\}) = -\langle \mathcal{T} e^{-\beta \hat{H}} O(x_1) O(x_2) O(x_3) \rangle, \quad \cdots$$

- Causal *[Kobes]* ✓

$$G_R(\{p\}) = G_{111} - e^{-\pi\omega_1} G_{211} - e^{-\pi\omega_2} G_{121} + e^{-\pi\omega_3} G_{221}$$
$$G_R(\{x\}) = -\theta(t_3 > t_2 > t_1) \langle e^{-\beta \hat{H}} [[O(x_3), O(x_2)], O(x_1)] \rangle$$
$$-\theta(t_3 > t_1 > t_2) \langle e^{-\beta \hat{H}} [[O(x_3), O(x_1)], O(x_2)] \rangle$$

Causal 3-Point Function

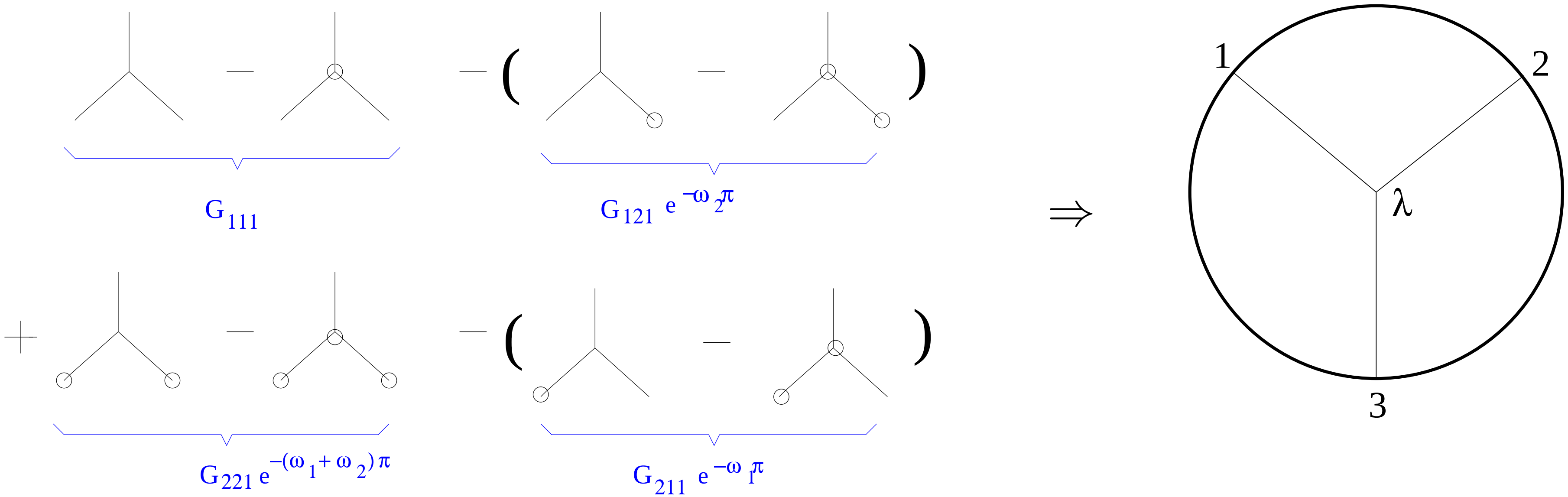
$$G_R(\{p\}) = G_{111} - e^{-\pi\omega_1} G_{211} - e^{-\pi\omega_2} G_{121} + e^{-\pi\omega_3} G_{221} = ?$$



$\Rightarrow$ **?**

Causal 3-Point Function

$$G_R(\{p\}) = G_{111} - e^{-\pi\omega_1}G_{211} - e^{-\pi\omega_2}G_{121} + e^{-\pi\omega_3}G_{221}$$



$$\Downarrow$$

$$G_R(\{p\}) = \frac{\lambda N^2}{16\pi^2 R^3}\delta^{(4)}(p_1+p_2-p_3)\int_0^1 du\sqrt{-g}F(p_1\,;u)F(p_2\,;u)F(p_3\,;u)$$

$$G_R(\{x\}) \sim \lambda \int dud^4y\sqrt{-g}\,F(x_1-y\,;u)F(x_2-y\,;u)F^*(x_3-y\,;u)$$

Conclusions & Outlook

- ***Conclusions***
  - Strongly-coupled FT $\rightarrow$ Weakly-coupled gravity
  - Finite-temperature $\rightarrow$ Black Hole in gravity
  - Mapping of everything: built & checked
  - 2-pt & 3-pt functions: calculated (in terms of Heun's function)
  - Causal correlators: very simple structures in gravity dual
- ***Future Work***
  - More realistic calculations (such as $T_{\mu\nu}$)
  - Non-relativistic, scale-invariant FT $\rightarrow$ gravity dual