Projection Hamiltonians for clustered quantum Hall wavefunctions

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Work in progress

Yale University

(brief) Talk outline

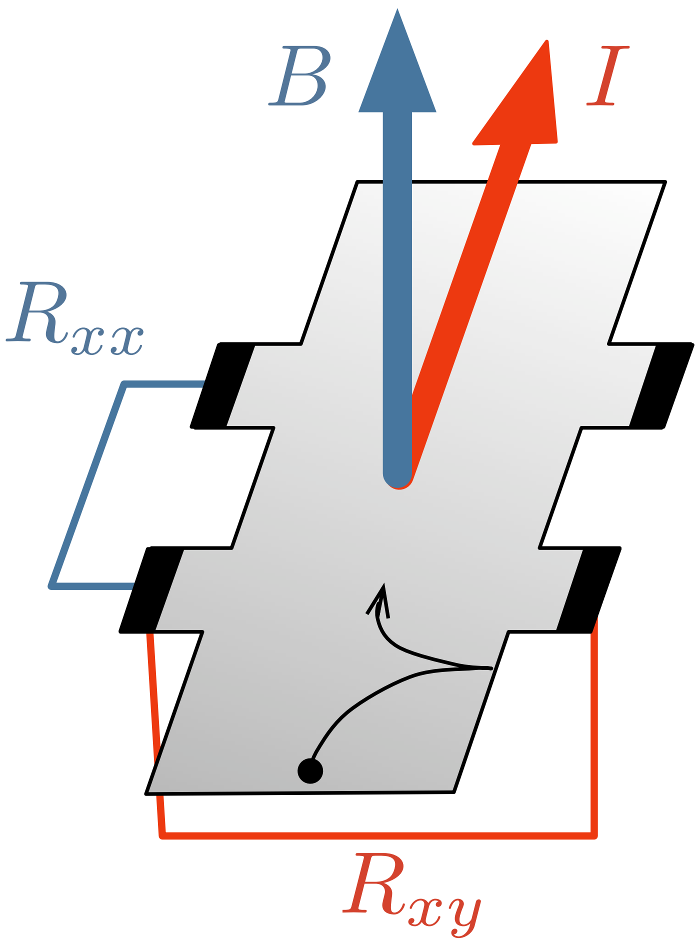
- Introduction to the theory of the fractional quantum Hall effect
  - Trial wavefunctions, projection Hamiltonians and conformal field theories (CFTs)
- Relating CFTs to Hamiltonians
  - Our example: three-body Hamiltonian and supersymmetric CFTs
  - Progress on obtaining Hamiltonians for a specific theory

Background on the quantum Hall effect

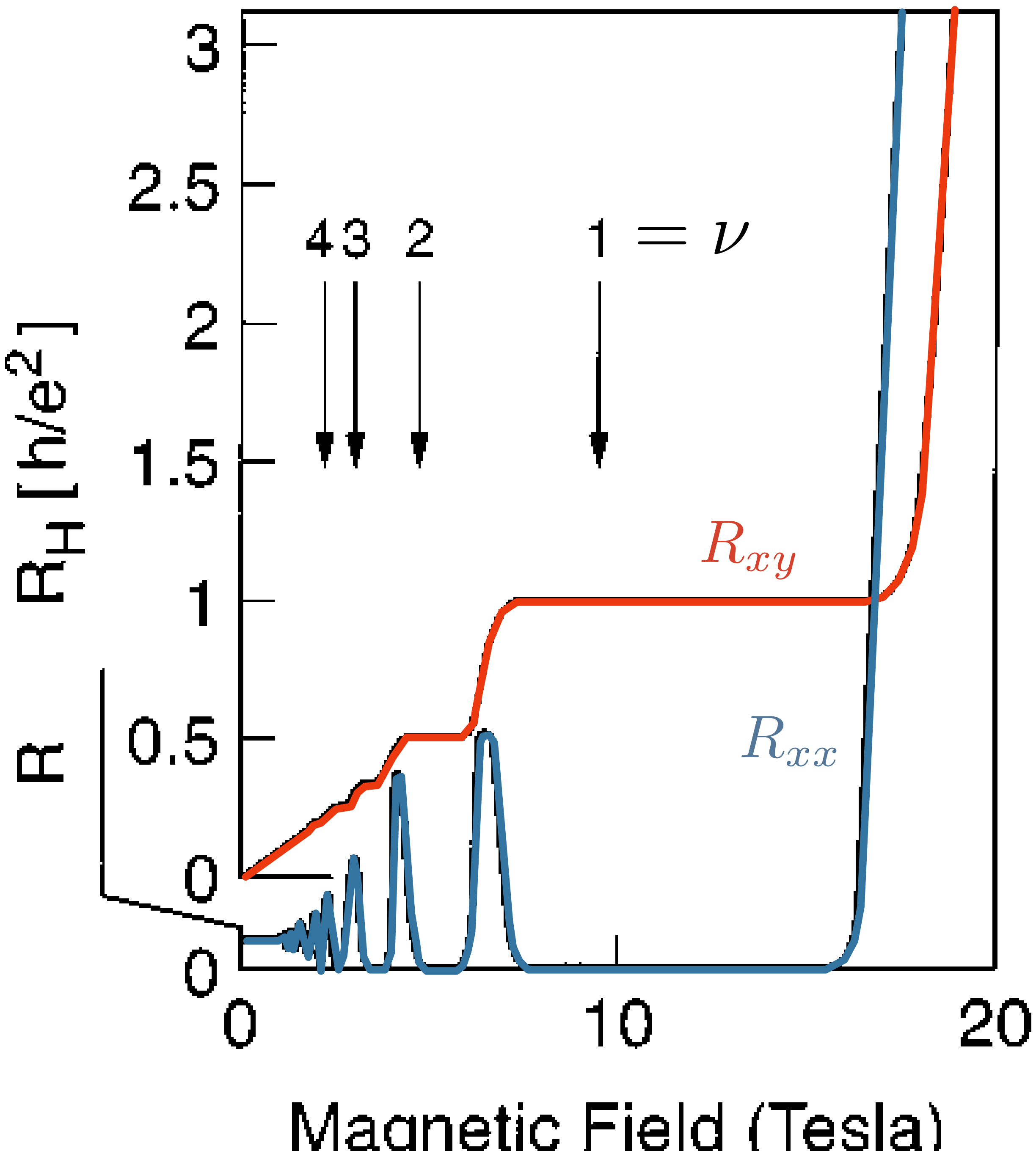
QHE: motivation

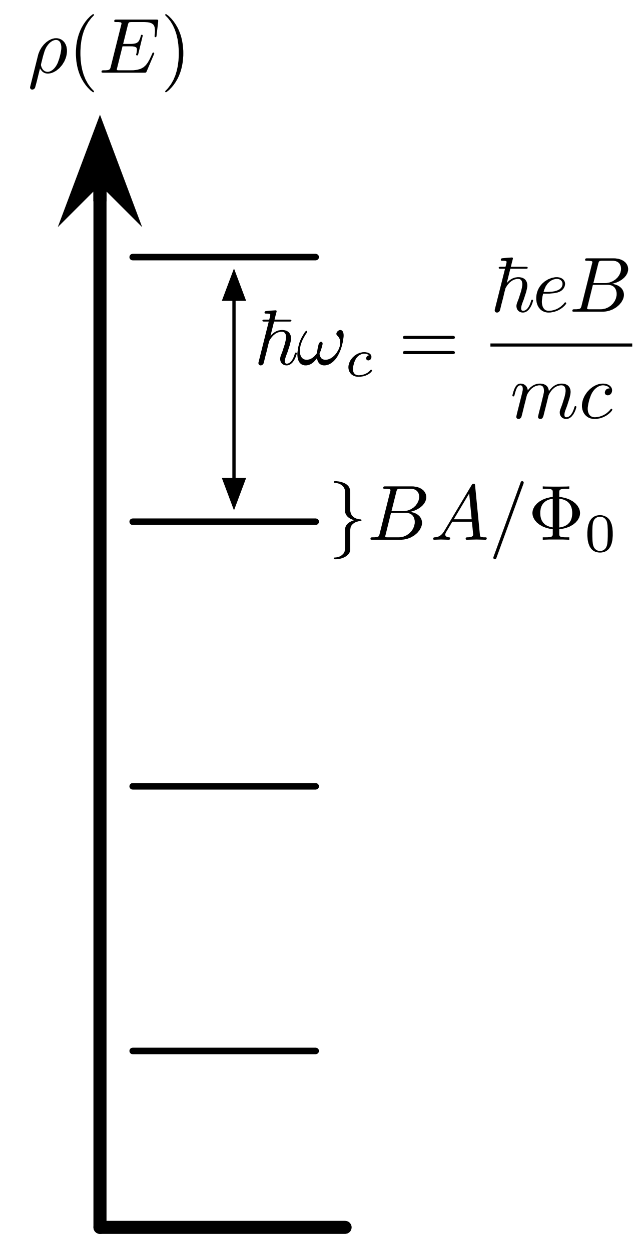
- Nobel prizes:
  - von Klitzing, 1985 — integer effect
  - Laughlin, Störmer, Tsui, 1998 — fractional effect
- *Topological* order: no local order parameter
- Realization of extended spin-statistics in d=2
- Application: topologically protected quantum computing

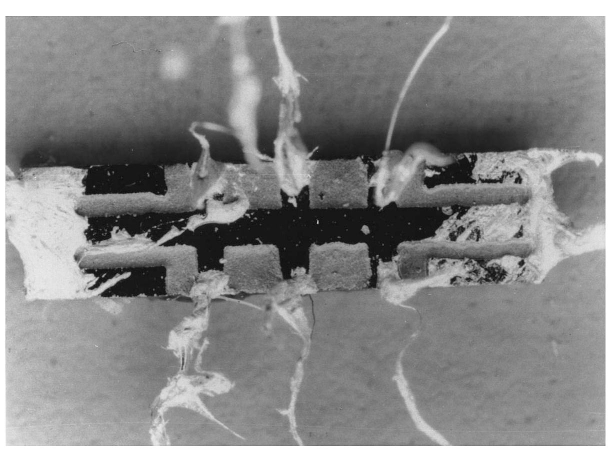
The quantum Hall effect



Landau level filling fraction $\nu = n_e \dfrac{\Phi_0}{B}$







From Stormer, RMP **71**, 875

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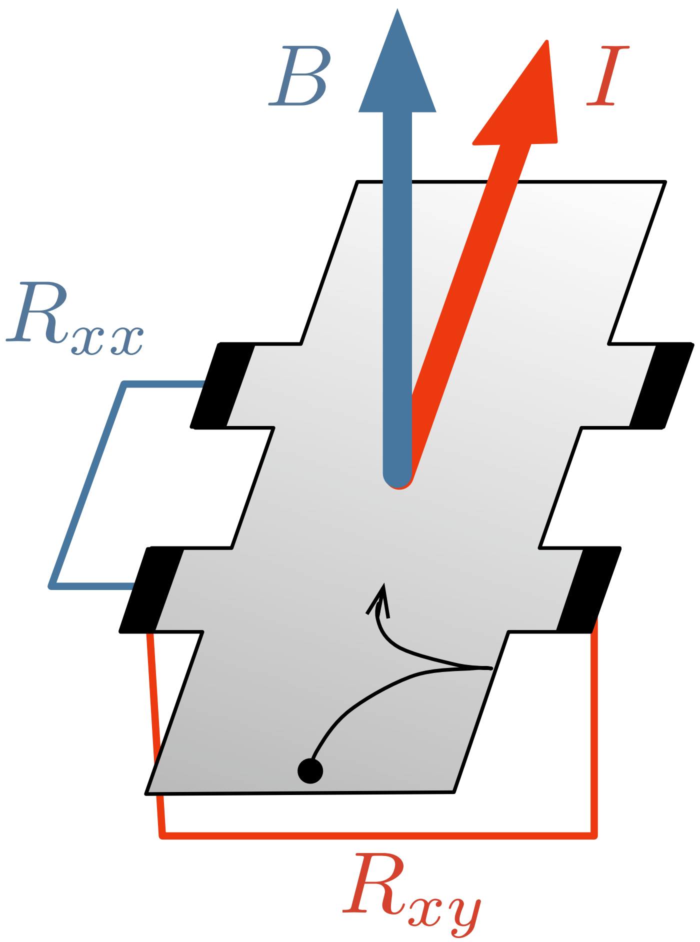

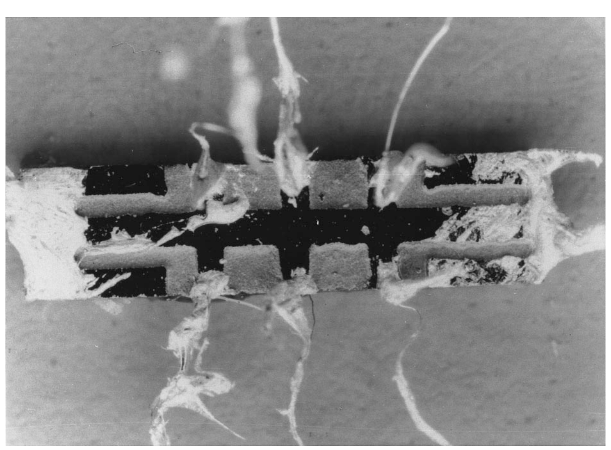

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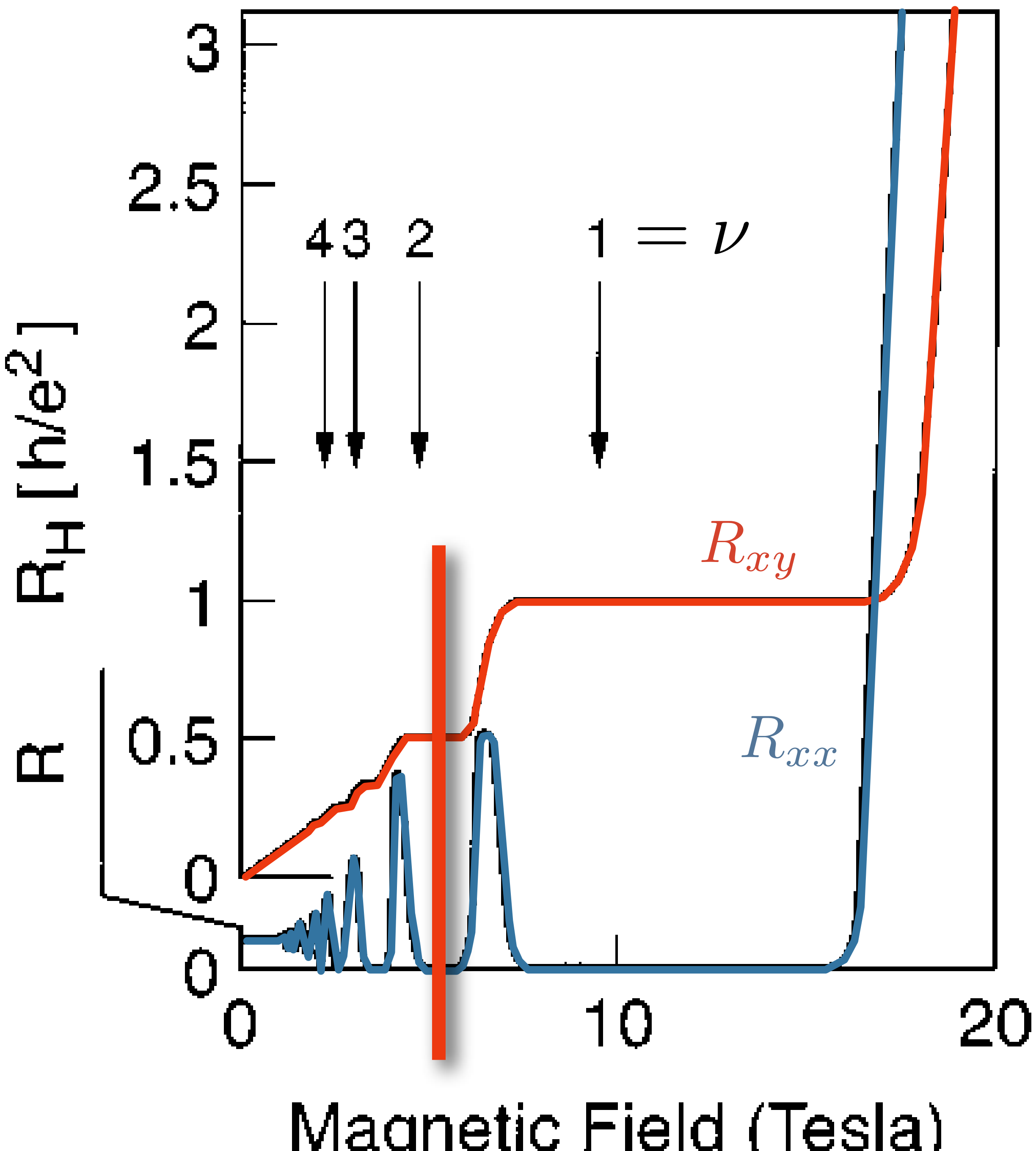

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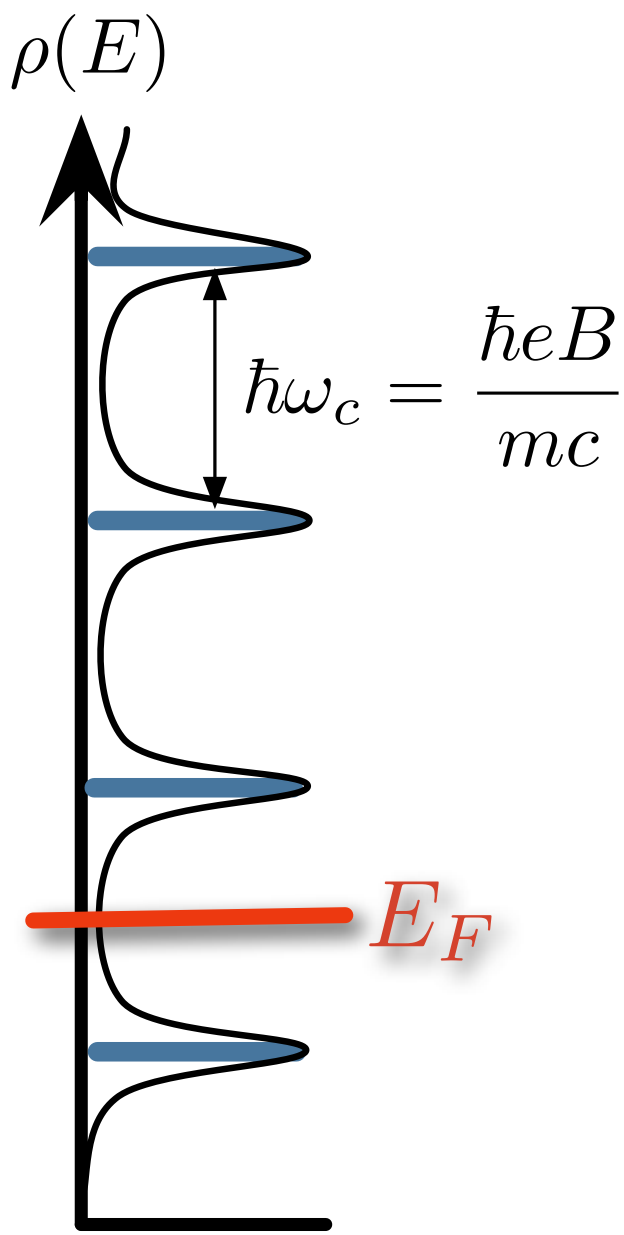

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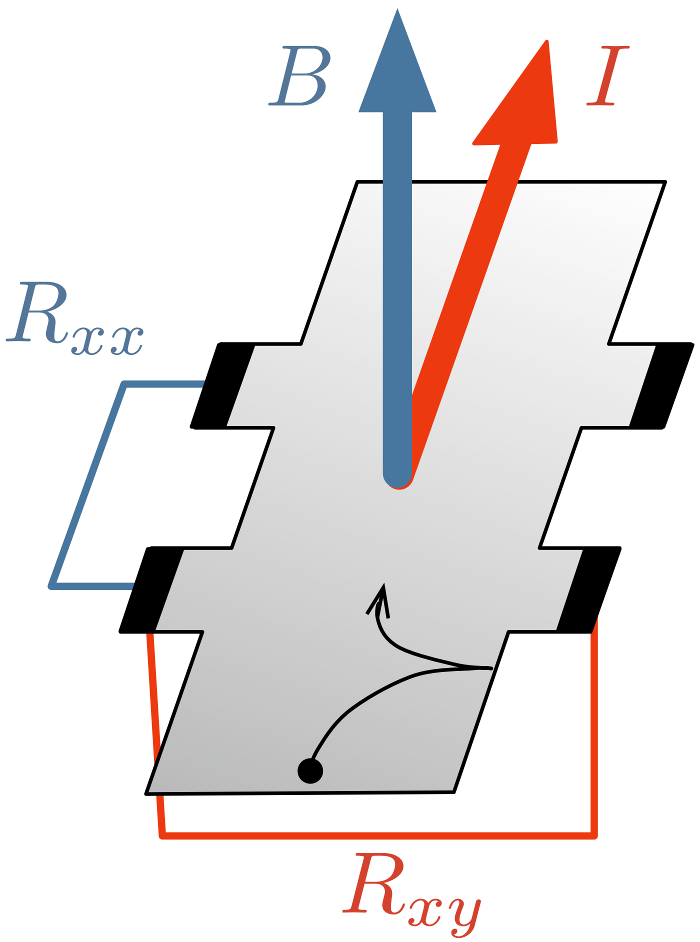

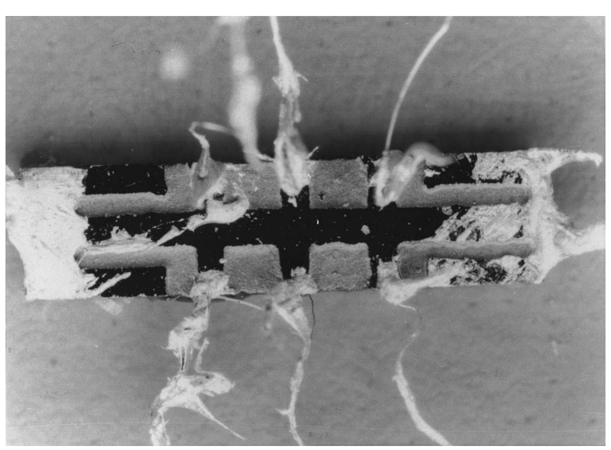

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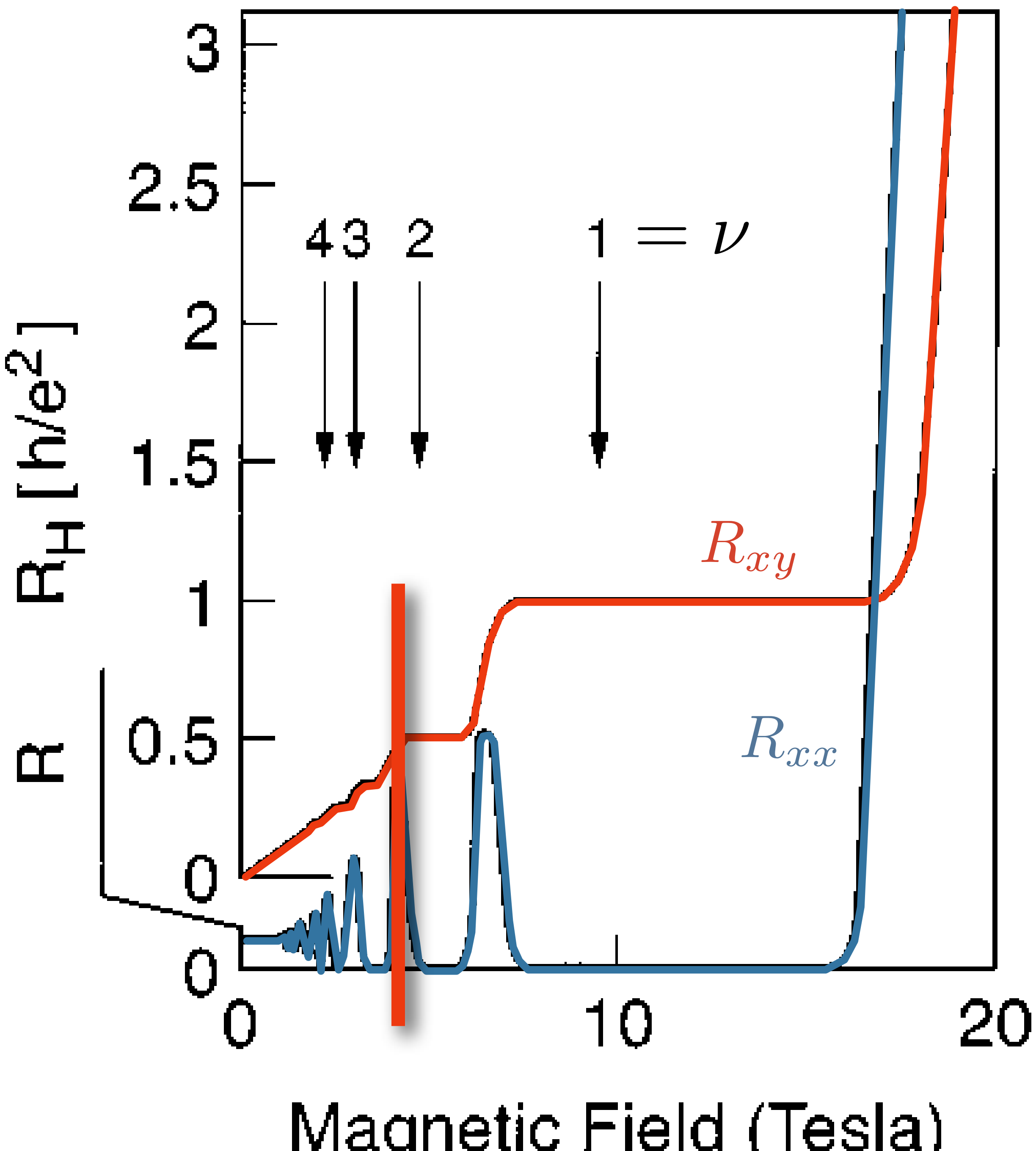

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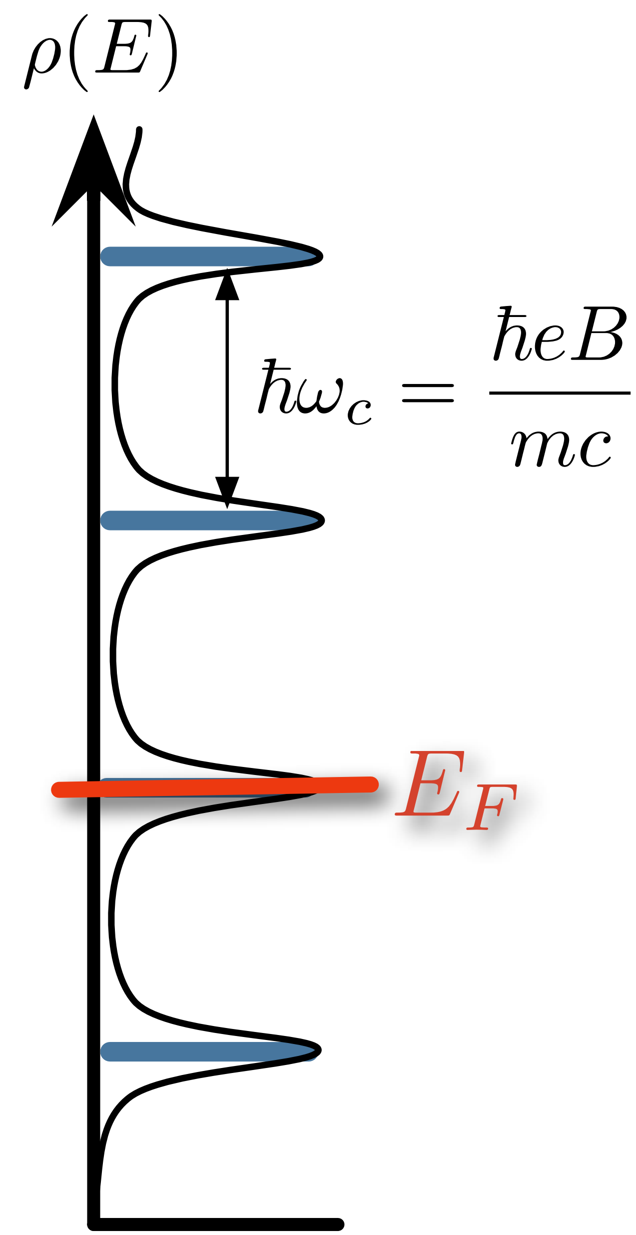

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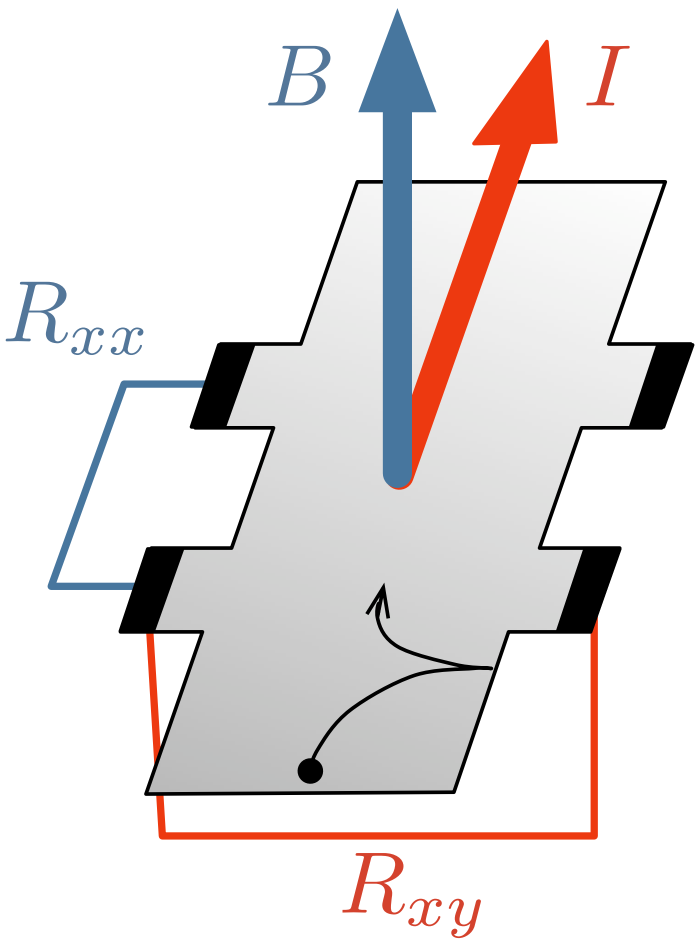
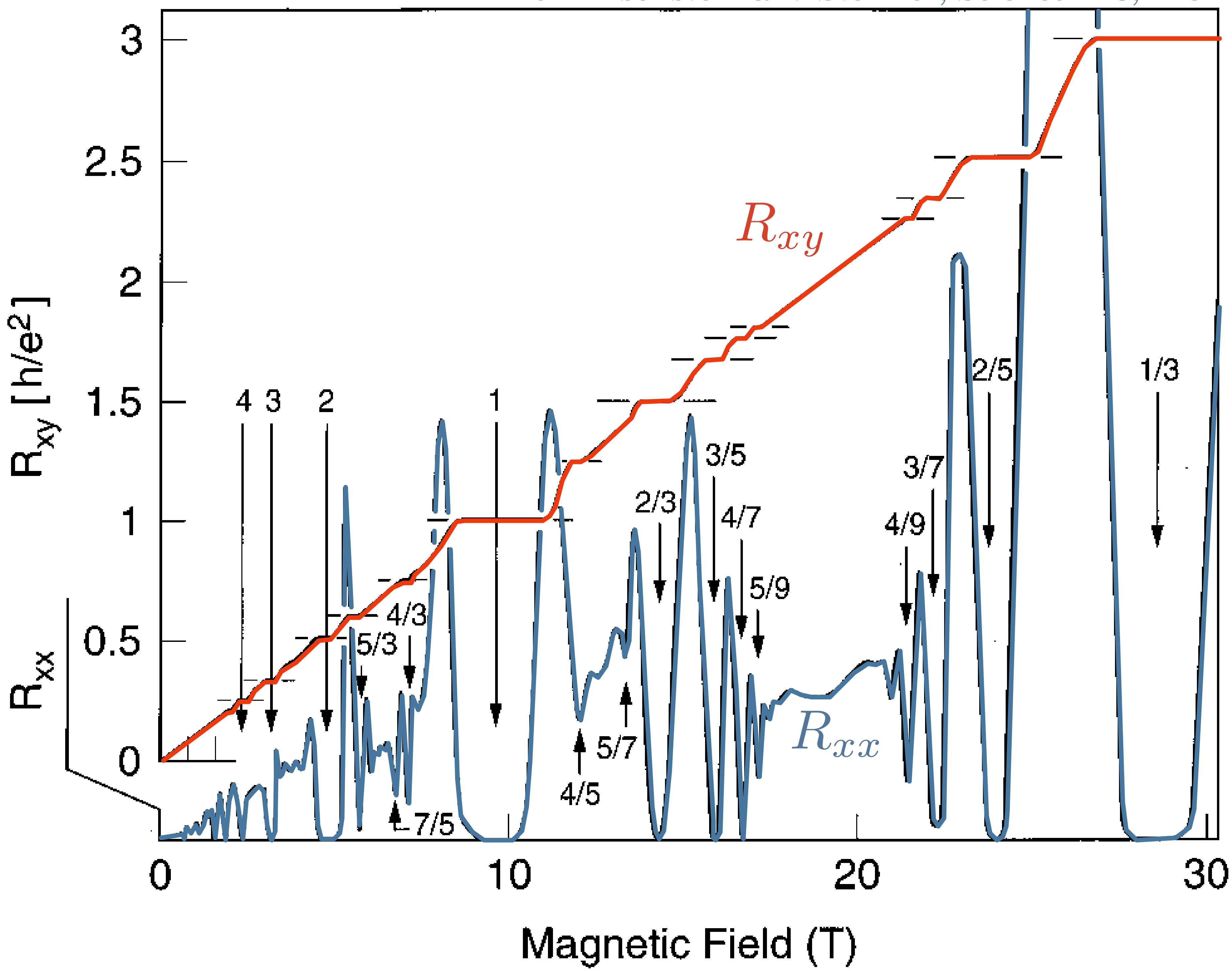
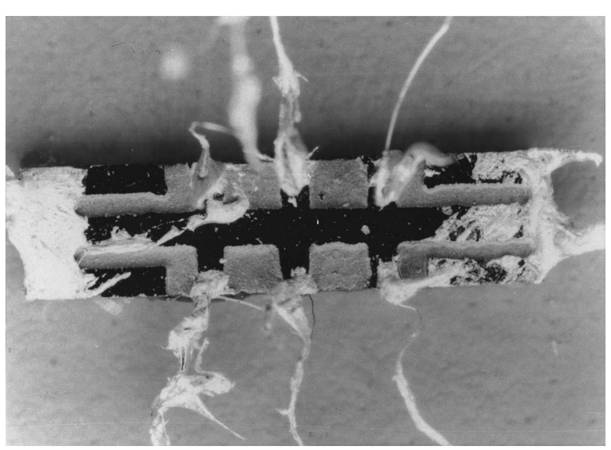


Landau level filling fraction $\nu = n_e \dfrac{\Phi_0}{B}$

From Eisenstein and Stormer, Science **248**, 1461





From Stormer, RMP **71**, 875

FQHE trial wavefunctions

FQH state is an *incompressible* electron fluid

FQH droplet (in plane) | Quasi-hole excitation | Quasi-electron excitation | (chiral) *edge* excitation

Symmetric gauge & lowest Landau level $\Rightarrow$ wavefunctions $\cong$ *analytic polynomials*

$$\psi_m(z) \propto \frac{z^m}{\ell_B^{m+1}} e^{-\frac{1}{4}\frac{|z|^2}{\ell_B^2}}$$

$$\Psi(z_1, \ldots, z_N) \propto \det\left[\psi_m(z_n)\right]_{m,n}$$

Problem: Landau levels are macroscopically degenerate; can't set up perturbation theory around non-interacting system!

FQHE trial wavefunctions

Problem: Landau levels are macroscopically degenerate; can't set up perturbation theory!

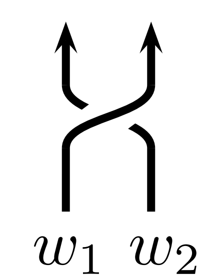
Laughlin: Account for Coulomb repulsion by extra Jastrow factors

$$\Psi_L(z_1, \ldots, z_N) \propto \underbrace{\prod_{i<j}(z_i - z_j)^m}_{J_N^m} \cdot e^{-\frac{1}{4}\sum_i |z_i|^2} \qquad \nu = \frac{1}{m}$$

(Validity established by exact diagonalization)

Quasihole: $\Psi_{L,w} \propto \prod_i (z_i - w) \cdot \prod_{i<j}(z_i - z_j)^m \cdot e^{-\frac{1}{4}\sum_i |z_i|^2}$

Quasiholes have *anyonic statistics:* braiding phase of $\theta = \pi/m$



Projection Hamiltonians

Rest of talk: FQHE of bosonic particles (*w/log*)
$\Rightarrow$ LLL Hilbert space $\cong$ *symmetric* polynomials

Haldane: Laughlin state is unique, exact highest-density eigenstate of projection Hamiltonian

Pseudopotentials

$$\mathcal{H} = \sum_{i<j} \sum_{\ell}^{1/\nu} V_\ell \boxed{\mathcal{P}_{i,j}[\ell]}$$

Basis:
$\left\{(z_i - z_j)^\ell\right\}$, ℓ *even*

$$\Psi(z_1, \ldots, z_N) = \sum_a \psi_a(z_1, z_2) \Psi'_a(z_3, \ldots, z_N)$$

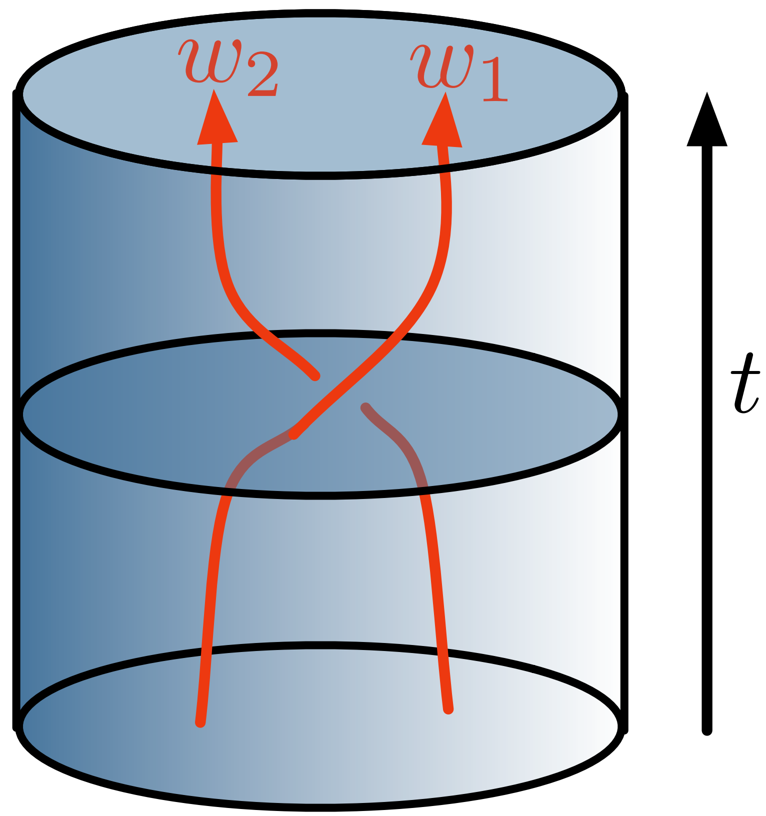
$$\psi_a(z_1, z_2) = \sum_{b,c} C_a^{b,c} \psi_b^{\mathrm{CM}} \left(\tfrac{1}{2}(z_1 + z_2)\right) \boxed{\psi_c^{\mathrm{rel}}(z_1 - z_2)} e^{-\frac{1}{4}(|z_1|^2 + |z_2|^2)}$$

Trial wavefunctions $\leftrightarrow$ Projection Hamiltonians

FQHE and CFT

Moore & Read: FQHE trial wavefunctions from “conformal blocks” of conformal field theory

Why? Fractional quasiparticle statistics
⇒ Chern-Simons TQFT
⇒ wavefunctions are
CFT amplitudes (Witten)



(Read: *d=1+1* edge excitation CFT same as *d=2+0* bulk wavefunction CFT)

Opens door for *non-Abelian* statistics!
(Willett, Pfeiffer & West): Experimentally observed?

"CFT for pedestrians"

Infinite number of local conformal transformations in d=*2* $\Rightarrow$ finite amount of data to specify theory

Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$

Rational theories have a finite number of primary fields (descendants may be *singular*)

$$|\phi\rangle$$
$$L_{-1}|\phi\rangle$$
$$L_{-2}|\phi\rangle,\ L_{-1}^2|\phi\rangle$$
$$L_{-3}|\phi\rangle,\ L_{-2}L_{-1}|\phi\rangle,\ L_{-1}^3|\phi\rangle$$
$$\dots$$

Operator product expansion (OPE)

$\bullet\, T$ $\bullet\, \phi$ $\bullet\, \phi'$

$$T(z)\phi(w,\overline{w}) \sim \frac{h\phi(w,\overline{w})}{(z-w)^2} + \frac{\partial\phi(w,\overline{w})}{z-w}$$

Fusion rules:

Ising CFT: ($c = 1/2$)

	Δ
ψ	$1/2$
σ	$1/16$

$\times$	ψ	σ
ψ	$\mathbf{1}$	
σ	σ	$\mathbf{1}+\psi$

$\Rightarrow$ *vector space* of conformal blocks $\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle$

Steps towards relating the CFT and Hamiltonian descriptions

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Trial wavefunctions $\leftrightarrow$ Projection Hamiltonians

Few-body Hamiltonians

Simon, Rezayi & Cooper: Systematic study of multiparticle pseudopotential Hamiltonians

$\Rightarrow$ Basis: translationally-invariant symmetric polynomials of degree ***r*** in ***k+1*** variables $\left(\nu = \frac{k}{r}\right)$

$$(k=2, r=2) : \tilde{e}_2 = z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_1 z_3$$

D(***k***,***r***) = Dimension of space spanned by these polynomials

$r =$	0	1	2	3	4	5	6	7	8	9	10	11	12
$k = 1$	1	0	1	0	1	0	1	0	1	0	1	0	1
$k =$ [illegible]	[illegible]	0	1	1	1	1	2	1	2	[illegible]	[illegible]	[illegible]	3
$k =$ [illegible]	[illegible]	0	1	1	2	1	3	2	4	[illegible]	[illegible]	[illegible]	7
$k =$ [illegible]	[illegible]	0	1	1	2	2	3	3	5	5	7	7	10
$k = 5$	1	0	1	1	2	2	4	3	6	6	9	9	14

Read-Rezayi

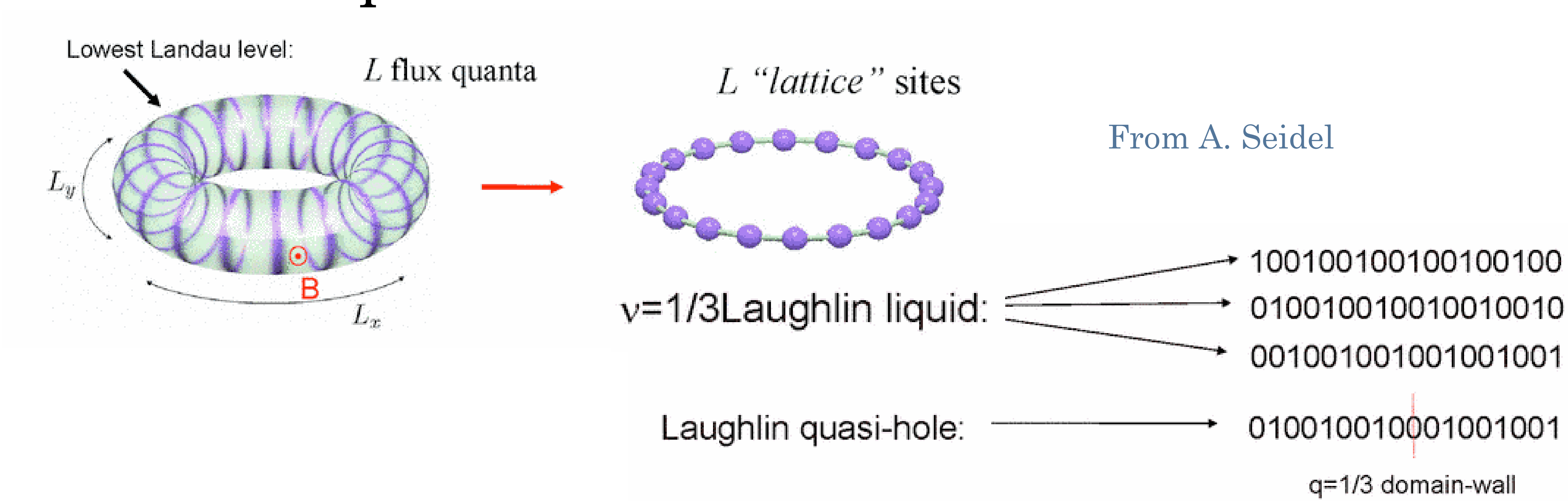
Laughlin

Q: What if the Hamiltonian penalizes all but one *k+1*-particle behavior at given *r*?

Hamiltonian will contain *continuous free parameters* selecting direction in subspace

Why? Important limitation of existing methods!

- Thin torus limit (Seidel, Lee *et. al.*, Bergholtz, Karlhede, Hansson, Hermanns *et. al.;* Ardonne): CDW orbital filling only specifies integer data; limit not unique



From A. Seidel

Q: What if the Hamiltonian penalizes all but one *k+1*-particle behavior at given *r*?

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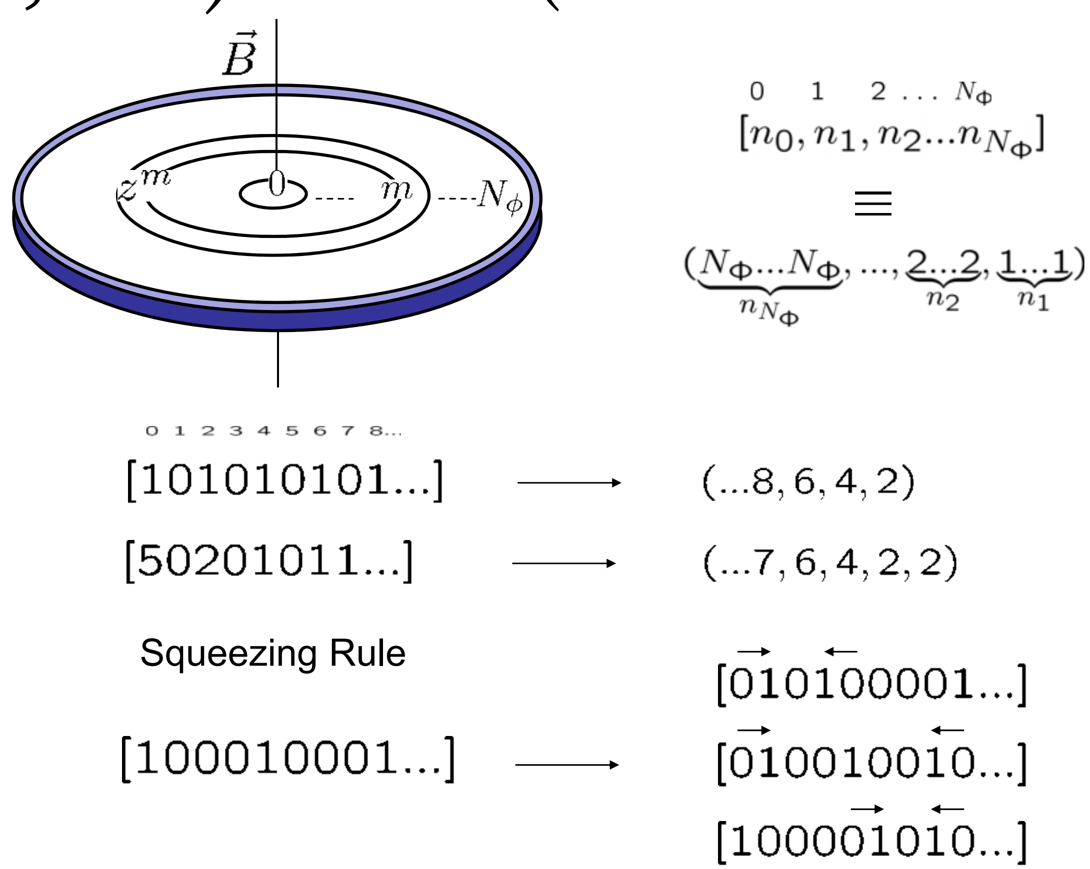
Why? Important limitation of existing methods!

- Jack polynomials (Bernevig, Haldane *et. al.*): (k,r) fix state for *single* Jacks; correspond to $M_k(k+1,k+r)$ CFTs (Estienne & Santachiara)



From Bernevig and Haldane, PRL **100**, 246802

Q: What if the Hamiltonian penalizes all but one *k+1*-particle behavior at given *r*?

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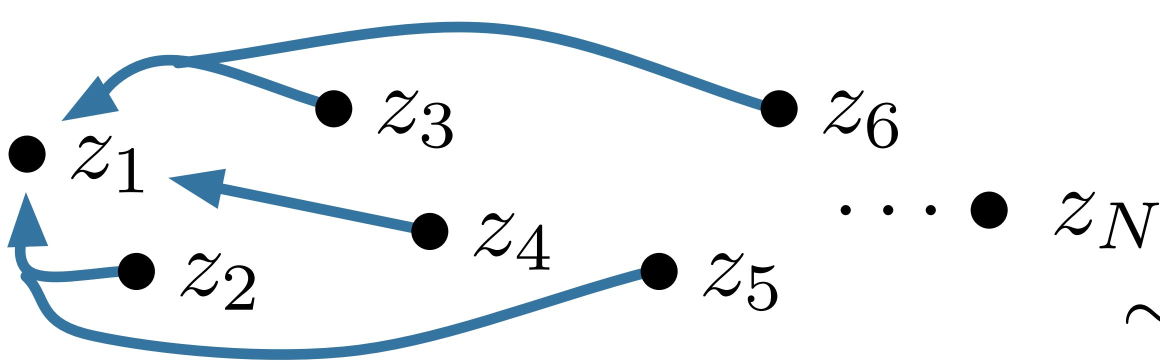
Why? Important limitation of existing methods!

- "Pattern of zeros" (Wen, Wang *et. al.*): also discrete; not unique and sufficient conditions not known; (later papers) additional CFT data must be added *a priori*



$$P(z_1^{(a)}, z_1^{(b)}, \cdots)\Big|_{z_1^{(a)} \to z_1^{(b)} \equiv z^{(a+b)}}$$
$$\sim (z_1^{(a)} - z_1^{(b)})^{\boxed{D_{ab}}} \tilde{P}(z^{(a+b)}, \cdots) + o((z$$

Q: What if the Hamiltonian penalizes all but one *k+1*-particle behavior at given *r*?

D(***k***,***r***):	0	1	2	3	4	5	6	7	8	9	10	11	12
	1	0	1	0	1	0	1	0	1	0	1	0	1
$k=2$	1	0	1	1	1	1	2	[illegible]	[illegible]	[illegible]	[illegible]	2	3
$k=3$	1	0	1	1	2	1	3	2	4	3	5	4	7
$k=4$	1	0	1	1	2	2	3	3	5	5	7	7	10

This work

Two linearly-independent ways for wavefunction to vanish as *r=6* powers as *k+1=3* particles coincide:

$$\widetilde{e}_2(z_1,z_2,z_3)^3 \propto \left(z_1^2+z_2^2+z_3^2-z_1z_2-z_1z_3-z_2z_3\right)^3$$

$$\widetilde{e}_3(z_1,z_2,z_3)^2 \propto (z_1+z_2-2z_3)^2\,(z_1-2z_2+z_3)^2 \times(-2z_1+z_2+z_3)^2$$

Connection to CFTs

$D(\boldsymbol{k},\boldsymbol{r})$:	0	1	2	3	4	5	6	7	8	9	10	11	12
	1	0	1	0	1	0	1					0	1
					1	1	2					2	3
					2	1	3					4	7
$k=4$	1	0	1	1	2	2	[illegible]	3	5	5	7	7	10

Simon, Rezayi & Regnault,
GS $\leftrightarrow$ S_3 CFTs

This work,
SCFTs

Simon: Supercurrent amplitudes at arbitrary c

$$\langle G(z_1)\cdots G(z_{2n})\rangle \propto J_{2n}^{-3}\,\mathcal{S}\left[\prod_{1\le i<j\le n}\chi_c(z_{2i-1},z_{2i};z_{2j-1},z_{2j})\right]$$

$$\chi_c(z_1,z_2;z_3,z_4) = 3z_{1,3}^4 z_{1,4}^2 z_{2,3}^2 z_{2,4}^4 + (c-3)z_{1,3}^3 z_{1,4}^3 z_{2,3}^3 z_{2,4}^3$$

Three-particle behavior:

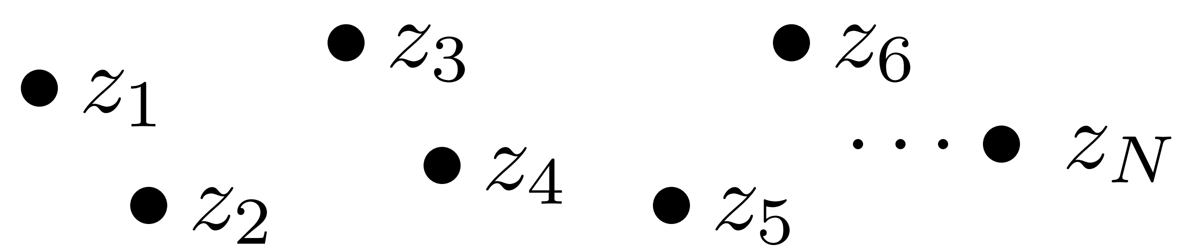
$$\mathcal{S}\left[\lim_{z_4\to\infty} z_4^{-6}\chi_c\right] = -(6+5c)\widetilde{e}_2(z_1,z_2,z_3)^3 + \left(-\frac{81}{2}+27c\right)\widetilde{e}_3(z_1,z_2,z_3)^2$$

SCFT wavefunctions

Obtain basis for *all* zero-energy edge excitations, explicitly, via filtration method (Ardonne, Kedem, Stone; Read)

Set of zero-energy edge excitations = *polynomial ideal* I_N

Clustering map C_m: make m clusters of k=2 particles



$$C_m \Psi(z_1, \ldots, z_N) =$$
$$\Psi(Z_1, Z_1, Z_2, Z_2, \ldots, Z_m, Z_m, z_{2m+1}, \ldots, z_N)$$

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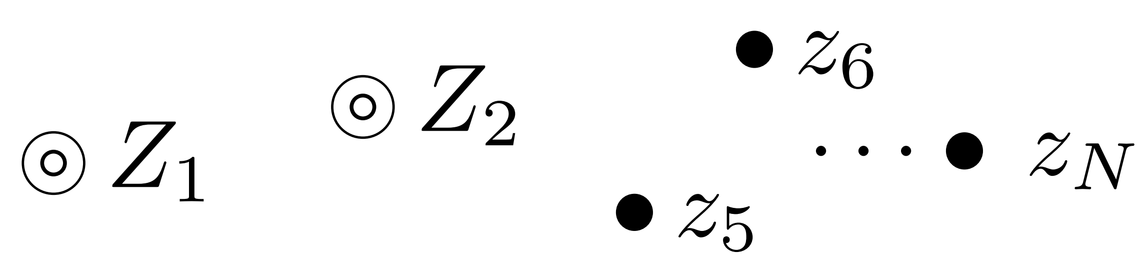

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$$\Psi(Z_1, Z_1, Z_2, Z_2, \ldots, Z_m, Z_m, z_{2m+1}, \ldots, z_N)$$

$$\mathrm{Im}\ C_m \cap I_N \propto \prod_{2m<i} \prod_{j \leq m} (z_i - Z_j)^6 \cdot \prod_{i<j\leq m} (Z_i - Z_j)^{12}$$

$$F_m = \ker\ C_m \cap I_N; \qquad F_0 = 0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{N/2} = I_N$$

SCFT wavefunctions

Obtain basis for *all* zero-energy edge excitations, explicitly, via filtration method (Ardonne, Kedem, Stone; Read)

$$F_m = \ker\, C_m \cap I_N; \qquad F_0 = 0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{N/2} = I_N$$

"residue"

$$\mathrm{Im}\; C_m|_{F_{m+1}}/\ker\, C_m \subseteq \prod_{2m<i}\prod_{j\leq m}(z_i - Z_j)^6 \cdot \prod_{i<j\leq m}(Z_i - Z_j)^{12}$$

Cluster-cluster and cluster-particle factors

$$\times \prod_{2m<i<j}(z_i - z_j)^2 \cdot I^{\mathrm{MR}}_{N-2m} \otimes \Lambda_m$$

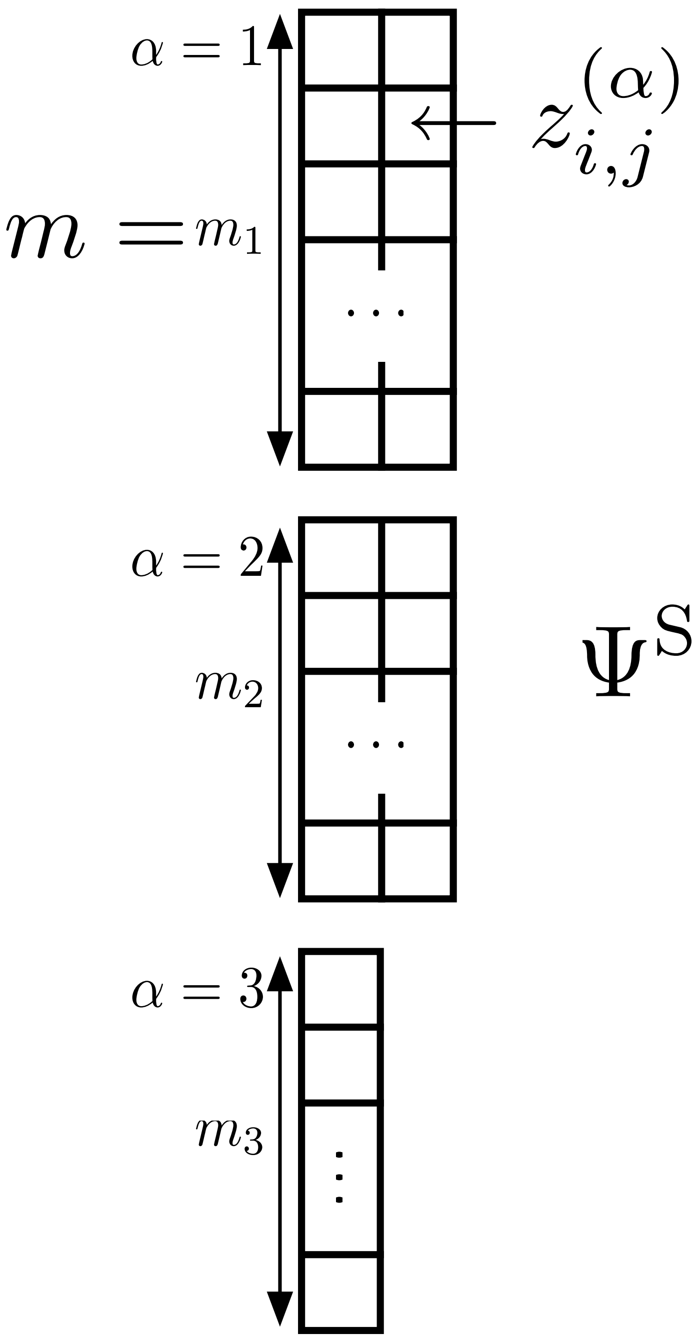
Irreducibility of three-body interaction

Charge excitations

Show this is an equality by construction of basis

SCFT wavefunctions

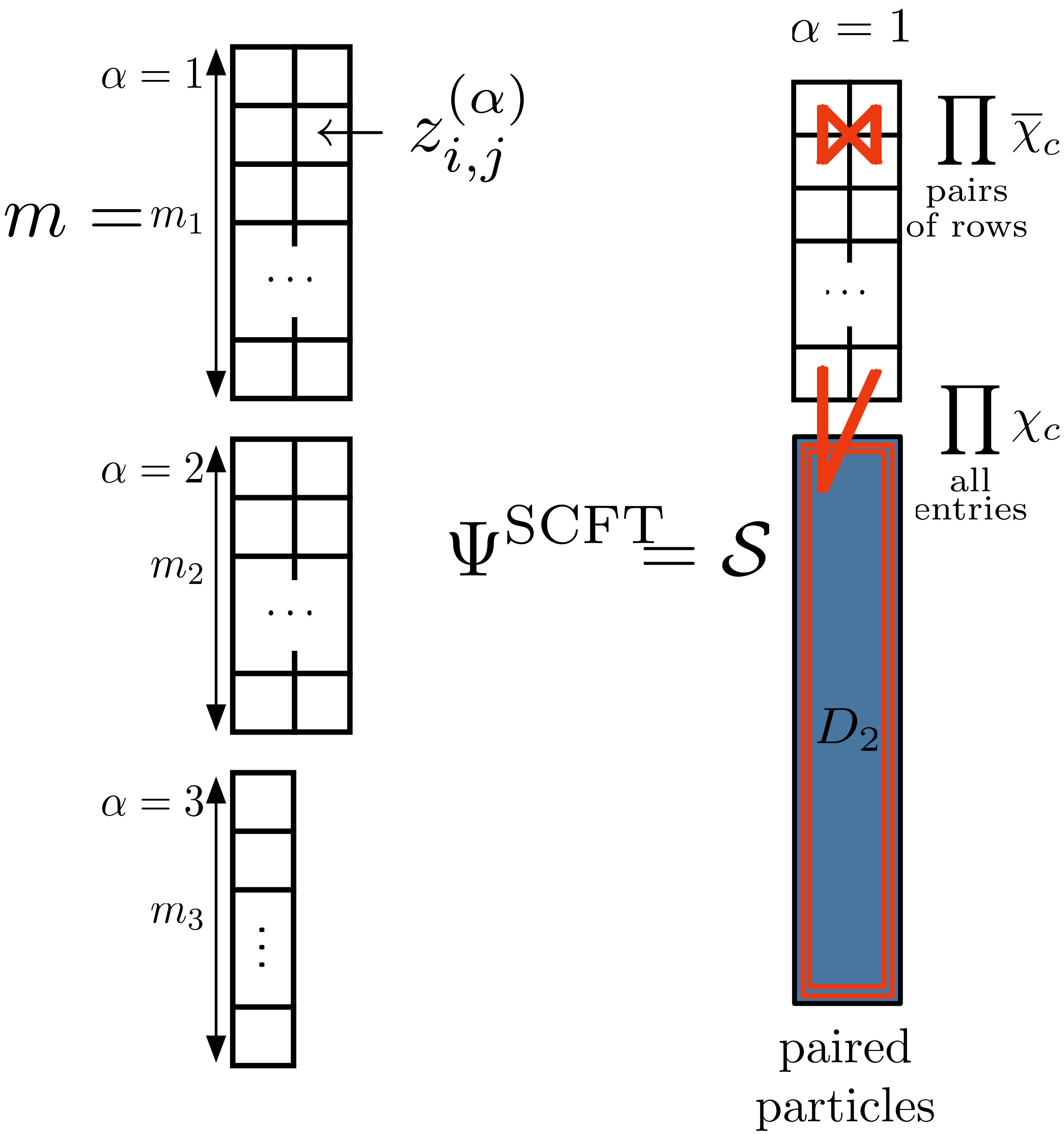
Obtain basis for *all* zero-energy edge excitations, explicitly, via filtration method (Ardonne, Kedem, Stone; Read)



$$\Psi^{\mathrm{SCFT}} =$$

SCFT wavefunctions

Obtain basis for *all* zero-energy edge excitations, explicitly, via filtration method (Ardonne, Kedem, Stone; Read)



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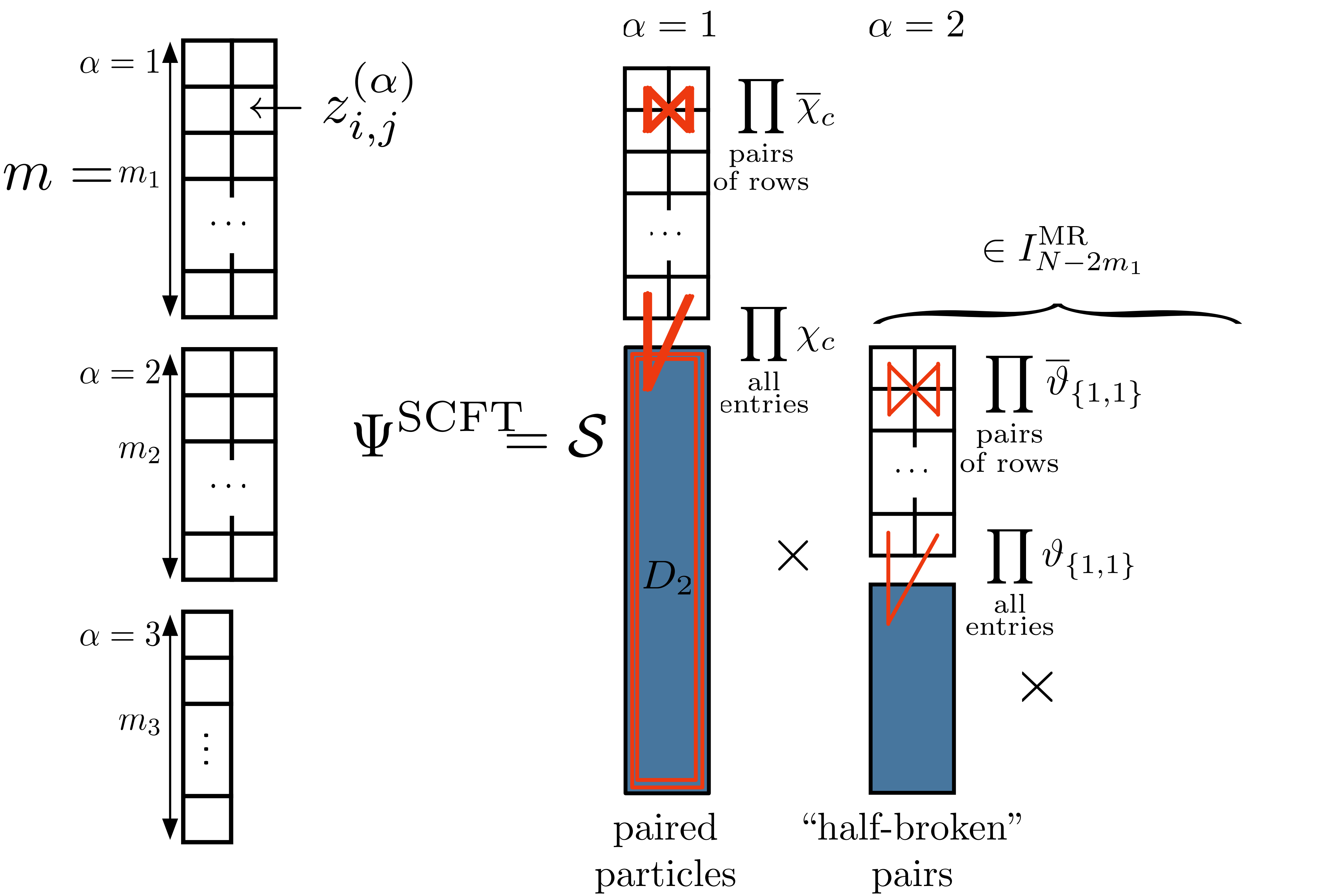

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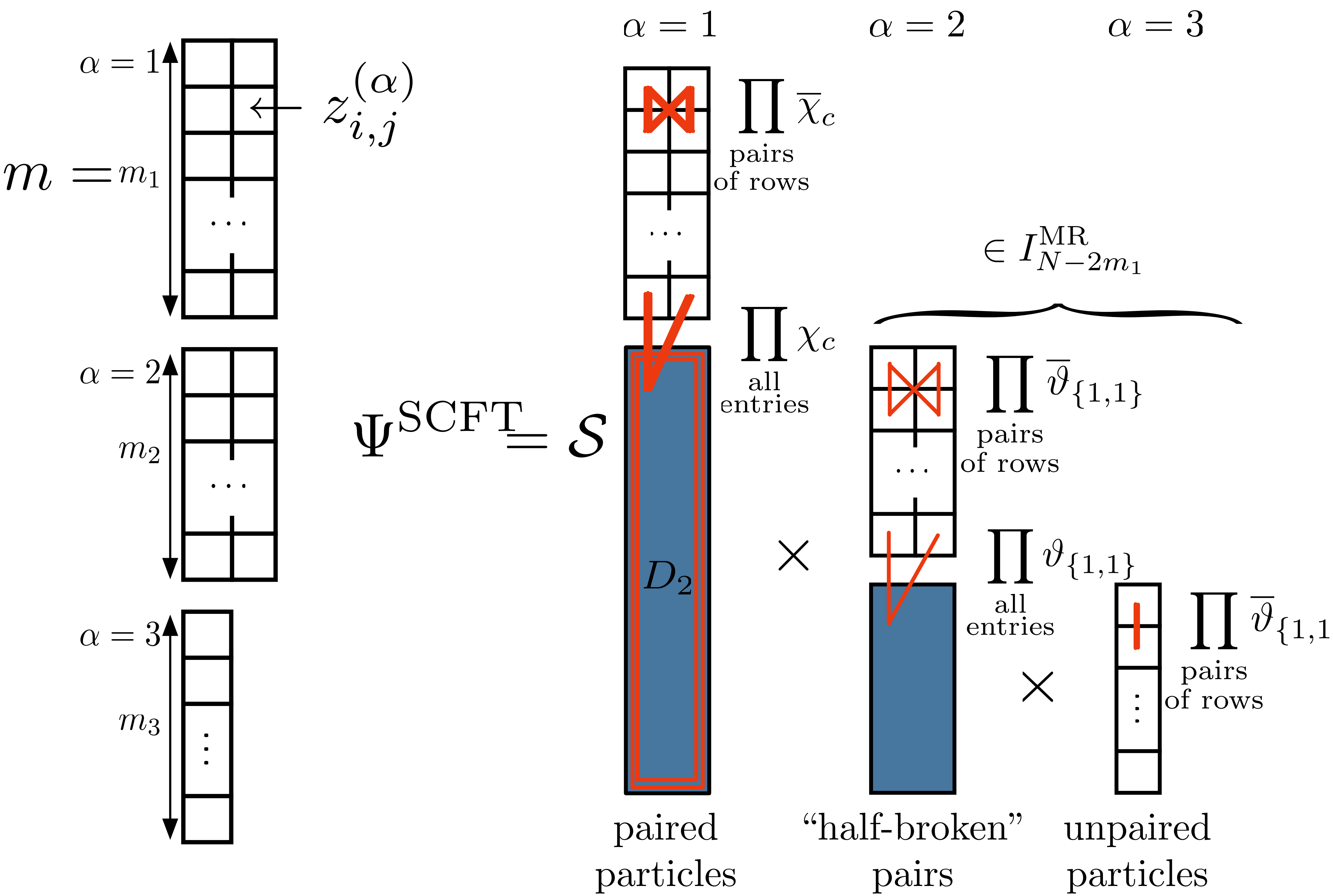

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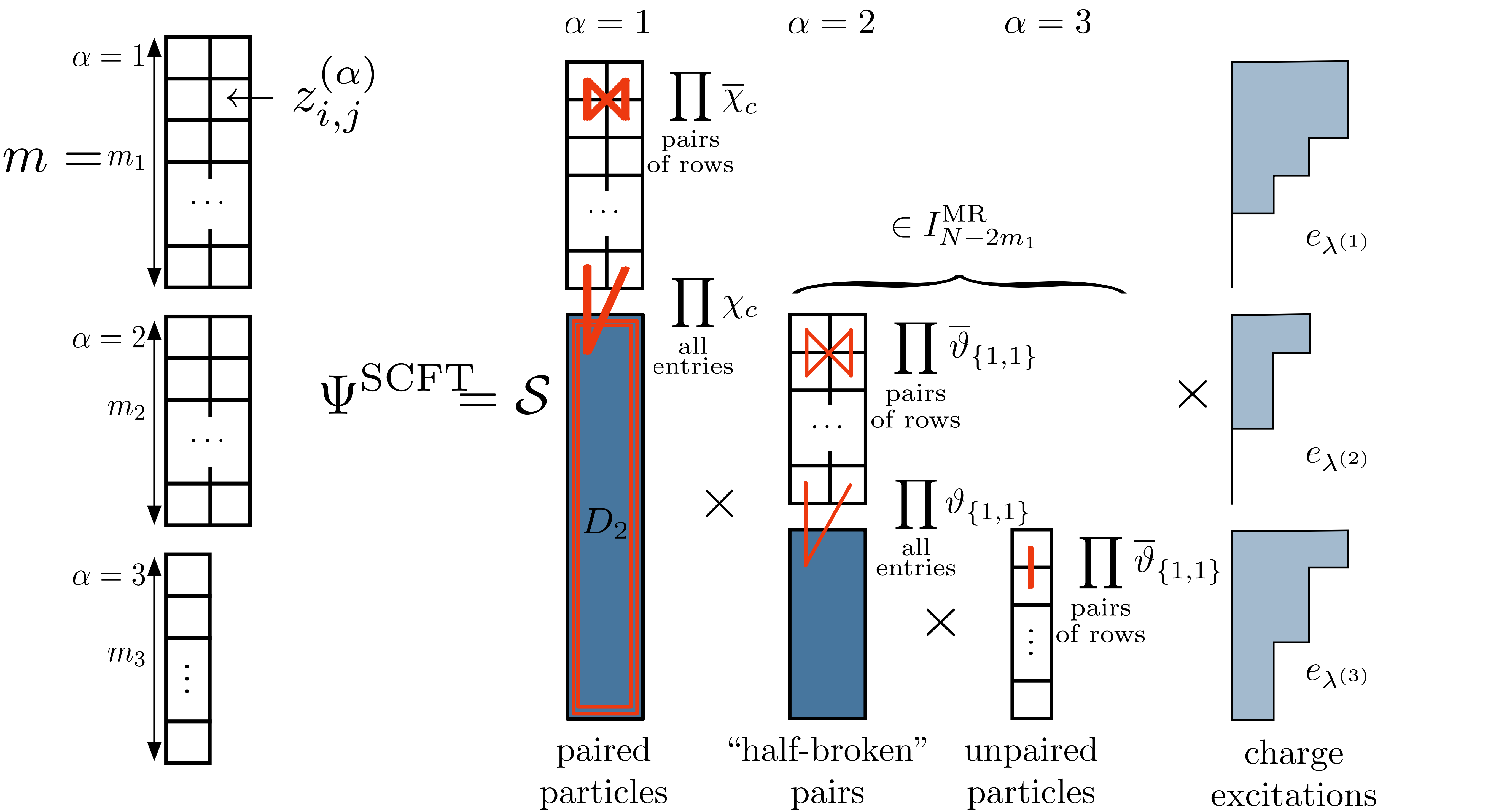

Counting wavefunctions

- # edge excitations at given angular momentum = character of edge excitation CFT (Wen)
- State counting gives character for *generic* SCFT — independent of *c* !

$$q^{-\frac{3}{2}N(N-2)}\underset{\textit{(plane)}}{\mathrm{ch}\ I_N} = \sum_{\substack{m_2,m_3\geq 0:\\ 2m_2+m_3\leq N,\\ (-1)^{m_3}=(-1)^N}} \frac{q^{2m_2+\frac{1}{2}m_3(m_3+2)}}{(q)_{\frac{1}{2}(N-2m_2-m_3)}(q)_{m_2}(q)_{m_3}}$$

$$\left((q)_m \equiv \prod_{k=1}^{m}(1-q^k)\right)$$

$$\lim_{N\to\infty}$$

$$\frac{1}{(q)_\infty}\chi^{\pm}_{\mathrm{Kac}} = \frac{(1-q)}{(1\pm q^{1/2})}\frac{\prod_{r=1}^{\infty}(1\pm q^{r-1/2})}{(q)^2_\infty}$$

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(plane)

$$\lim_{N\to\infty}$$

$$\frac{1}{(q)_\infty}\chi^{\pm}_{\mathrm{Kac}} = \frac{(1-q)}{(1\pm q^{1/2})}\frac{\prod_{r=1}^{\infty}(1\pm q^{r-1/2})}{(q)_\infty^2}$$

- Generic SCFT nonrational $\Rightarrow$ Hamiltonian is gapless for all c

Recap

Hamiltonians

CFTs

(k,r) do not uniquely specify clustered state

SCFT blocks interpolate between (k=2,r=6) behaviors as function of c

Generated basis for edge excitations of (k=2,r=6,c) Hamiltonian

Edge theory is generic SCFT, for any value of c in Hamiltonian

Hamiltonian is gapless for all c

Want gapped (stable) state = unitary, rational CFT (Read); project out singular vectors

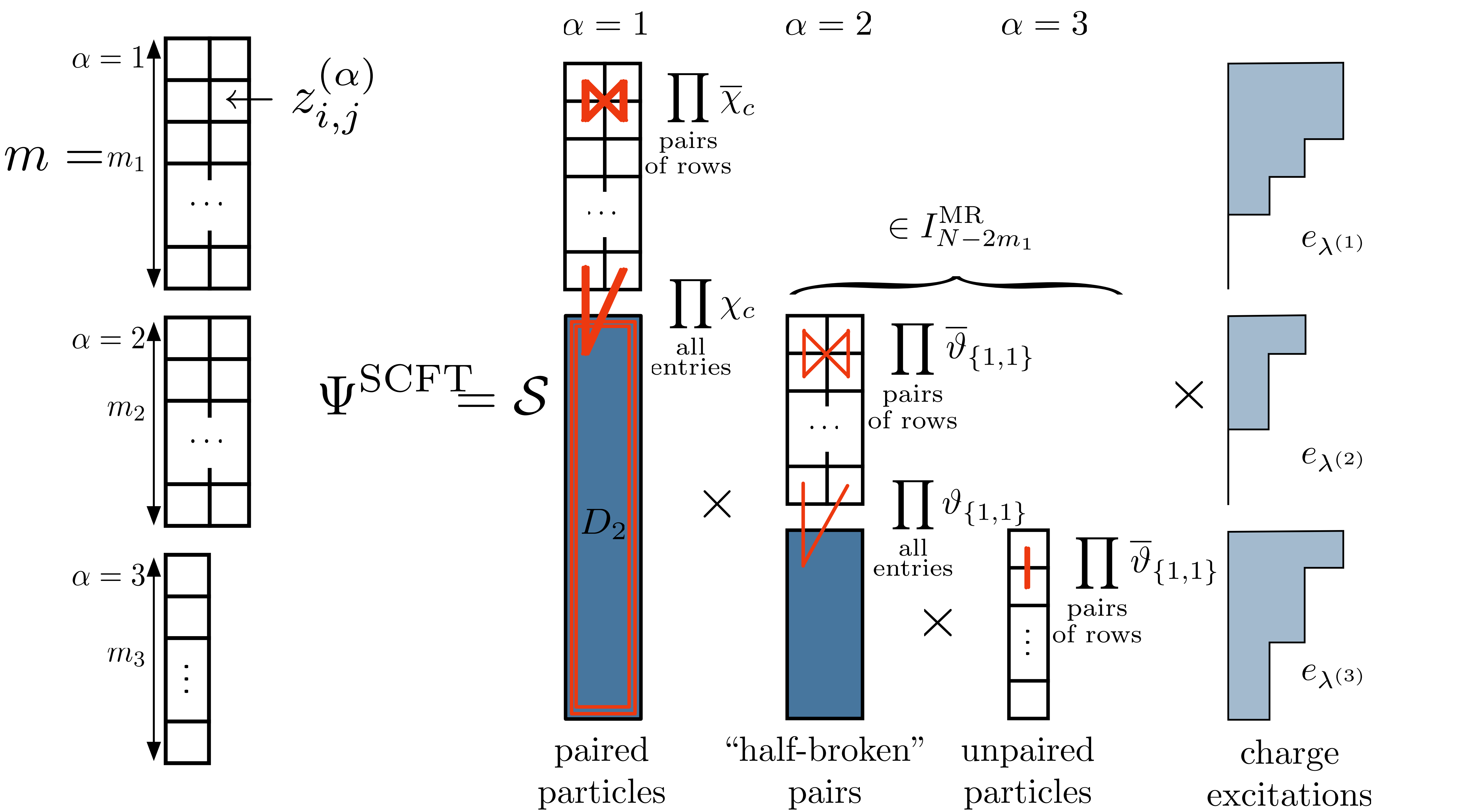
“Improved” Hamiltonians and rational SCFTs

“Improving” the Hamiltonian

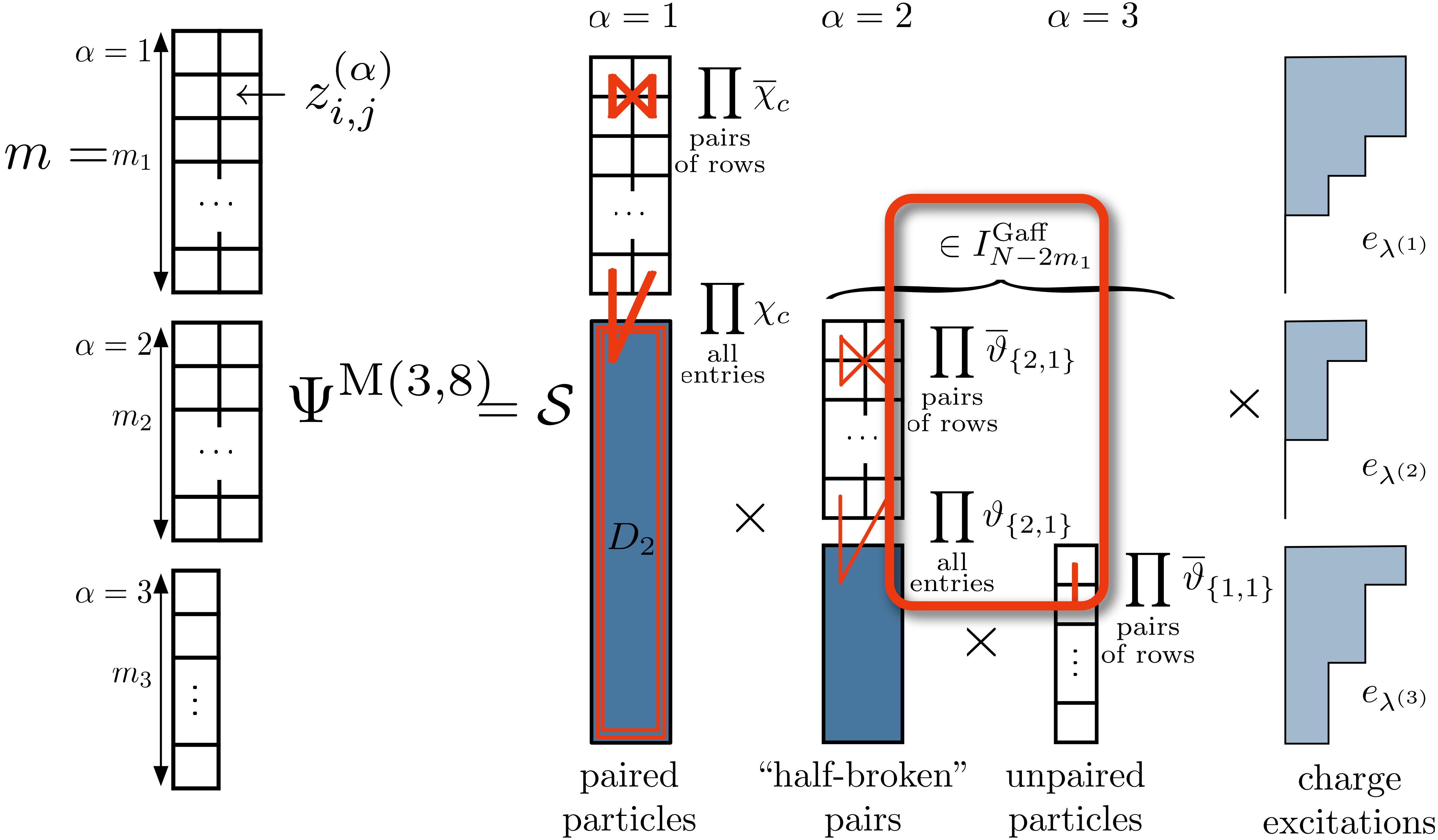
Obtain (unitary) minimal SCFTs by projecting out additional states: How many? Which ones?

- Use results of Feigin, Jimbo & Miwa for Virasoro M(3,p), a.k.a. $k = 2$ series of Jacks
  - Only three-body constraints (interactions) required
  - Recursive structure: polynomial ideal of zero-energy wavefunctions for M(3,p) related to that of M(3,p-3)
- Completely solved instance: SM(2,8) = M(3,8)
  - Project out additional three-particle state at degree 8
  - Manifest as extra couplings between “half-broken” excited pairs (built from Gaffnian wavefunctions)

SM(2,8) wavefunctions



SM(2,8) wavefunctions



SM(2,8) wavefunctions

Confirm basis is correct: recover known character for SM(2,8)

$$\frac{1}{(q)_\infty}\hat{\chi}^{\pm}_{\mathrm{Kac}} = \sum_{\substack{m_2,m_3\geq 0:\\ (-1)^{m_3}=\pm 1}} \frac{q^{2m_2+\frac{1}{2}m_3(m_3+2)}}{(q)_\infty (q)_{m_2} (q)_{m_3}}$$

$$\downarrow$$

$$\frac{1}{(q)_\infty}\hat{\chi}^{[2,8]}_{1,1} = \sum_{m_2,m_3\geq 0} \frac{q^{m_2^2+m_2m_3+\frac{1}{2}m_3^2+m_2+m_3}}{(q)_\infty (q)_{m_2} (q)_{m_3}}$$

Hamiltonian: need to project out *one* additional behavior at degree eight (geometry-dependent)

Keep behavior $\propto 9\widetilde{e}_3(z_1,z_2,z_3)^2\widetilde{e}_2(z_1,z_2,z_3) - \widetilde{e}_2(z_1,z_2,z_3)^4$

In plane, remove behavior $\propto 54\widetilde{e}_3(z_1,z_2,z_3)^2\widetilde{e}_2(z_1,z_2,z_3) + 11\widetilde{e}_2(z_1,z_2,z_3)^4$

Other minimal SCFTs?

- Would like a *unitary* minimal SCFT, *i.e.* gapped, stable FQH state
- These appear to require significant modifications to our formalism!
  - Simplest ex: Tricritical Ising model, SM(3,5) = M(4,5)
  - Manual construction of Verma module $\Rightarrow$ must have *seven*-particle interaction: clusters of clusters?

$$\frac{1}{(q)_\infty}\hat{\chi}^{[3,5]}_{1,1} = \sum_{\vec{m}\geq 0} \frac{q^{\vec{m}A\vec{m}}}{\prod_{i=1}^{7}(q)_{m_i}} \qquad A = \begin{pmatrix} \frac{3}{2} & 1 & \frac{3}{2} & 2 & 2 & \frac{5}{2} & 3 \\ 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ \frac{3}{2} & 2 & \frac{7}{2} & 3 & 4 & \frac{9}{2} & 6 \\ 2 & 2 & 3 & 4 & 4 & 5 & 6 \\ 2 & 3 & 4 & 4 & 6 & 6 & 8 \\ \frac{5}{2} & 3 & \frac{9}{2} & 5 & 6 & \frac{15}{2} & 9 \\ 3 & 4 & 6 & 6 & 8 & 9 & 12 \end{pmatrix}$$

$$\frac{1}{(q)_\infty}\hat{\chi}^{[3,5]}_{1,1} = \sum_{\substack{n_1,n_2\geq 0: \\ n_2\geq n_1}} \frac{q^{\frac{1}{2}n_1^2+2n_2^2-n_1n_2}(q)_{n_2}}{(q)_\infty (q)_{2n_2}(q)_{n_1}(q)_{n_2-n_1}}$$

Summary

- Take-home points:
  - (k,r) alone are *insufficient* to uniquely specify a clustered quantum Hall state
  - Must consider *entire* zero-energy edge excitation spectrum before identifying eigenspace of Hamiltonian with CFT, not just densest (ground) state
- This work (so far):
  - Constructed and counted complete set of zero-energy eigenstates of projection Hamiltonian with continuous free parameter
  - Found complete set of states and modified Hamiltonian corresponding to SM(2,8) minimal SCFT, a.k.a. $(k=2,r=6)$ Jack state