Geometrical Interpretation of Casimir Energy: The Case of Infinite Cylindrical Wedge

Hussain Zaidi

University of Virginia

March 17, 2009

Outline

Casimir Effect

Casimir effects are manifestations of the change in vacuum zero-point energy due to the introduction of boundaries.

The Vacuum

Electromagnetic Hamiltonian:

$$\mathcal{H} = \frac{1}{8\pi} \int \left(E^2 + B^2\right) d^3x$$

Quantized $\mathcal{H}$ is a sum of simple harmonic oscillators:

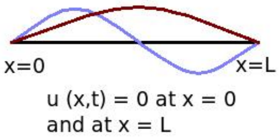
$$\mathcal{H} = \sum_m \hbar\omega_m \left(a_m^\dagger a_m + \frac{1}{2}\right)$$

- Vacuum ground state: $E_0 = \frac{1}{2} \sum_m \hbar\omega_m$
- Boundary conditions determine allowed ω_m

Elementary Ground State Calculation

Consider a one dimensional interval of length L (Dirichlet):

$$\omega_n = \frac{\pi c}{L} n \Rightarrow E_0 = \frac{\pi \hbar c}{2L} \sum_{n=1}^{\infty} n$$



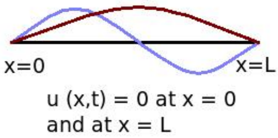
Regularize: $$E_0 = \frac{1}{2} \sum_m \hbar \omega_m e^{-(\omega_m/\omega_0)}$$

For 1D case: $$E_0 = \frac{\pi \hbar c}{2L} \left(+ \frac{\omega_0^2 L^2}{\pi^2 c^2} - \frac{1}{12} \right)$$

Elementary Ground State Calculation

Consider a one dimensional interval of length L (Dirichlet):

$$\omega_n = \frac{\pi c}{L} n \Rightarrow E_0 = \frac{\pi \hbar c}{2L} \sum_{n=1}^{\infty} n$$



Regularize:

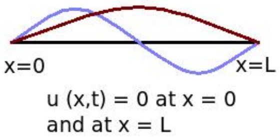
$$E_0 = \frac{1}{2} \sum_m \hbar \omega_m e^{-(\omega_m/\omega_0)}$$

For 1D case: $E_0 = \frac{\pi \hbar c}{2L} \left(+ \frac{\omega_0^2 L^2}{\pi^2 c^2} - \frac{1}{12} \right)$

Elementary Ground State Calculation

Consider a one dimensional interval of length L (Dirichlet):

$$\omega_n = \frac{\pi c}{L} n \Rightarrow E_0 = \frac{\pi \hbar c}{2L} \sum_{n=1}^{\infty} n$$



Regularize:

$$E_0 = \frac{1}{2} \sum_m \hbar \omega_m e^{-(\omega_m/\omega_0)}$$

For 1D case: $E_0 = \frac{\pi \hbar c}{2L} \left(+ \frac{\omega_0^2 L^2}{\pi^2 c^2} - \frac{1}{12} \right)$

1948: Discovery of the Casimir Effect

Casimir's Prediction: Two parallel perfectly conducting metal plates in vacuum attract each other:

$$\frac{E_0}{A} = -\frac{\pi^2}{720}\frac{\hbar c}{a^3}$$

$$\frac{F}{A} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4} \sim \frac{-10^{-27} Nm^2}{a^4}$$

Figure: H.B.G. Casimir
www.absoluteastronomy.com

- Dimensional analysis fixes $\frac{E_0}{A} \sim \frac{hc}{a^3}$
- Attractive force, independent of cut off ω_0

Could Casimir forces be responsible for stabilizing an electron?

Areas of Interest

- Sign of the Casimir Energy as a function of geometry
- Sonoluminescence - Dynamic Casimir effect
- Casimir Force for Nanoelectronics
- **Physical Interpretation of Non-Universal Terms**

Physical Interpretation of Non-universal Terms

- What is the significance of the cut off in terms of material properties?
- What is the physical origin of non-universal terms?

1980 Sen: Significance of the cut off

Casimir energy of a perfectly conducting ring in 2D using an exponential cut off:

$$E = \frac{0.045}{2R} + \frac{S\omega_0^2}{4\pi} - \frac{\ln(\omega_0 R)}{256}\int c^2(s)ds$$

- Physical justification for the cut off: real materials become transparent to high energy modes above some frequency ω_0.

Origin of Non-Universal Terms: 1979 - Deutch and Candelas

$$E \propto S \int \omega^2 d\omega + \int_{\partial M} \chi ds \int \omega d\omega$$

Origin of Non-Universal Terms: 2002 - Graham, Jaffe et. al.

Non-universal terms are due to artificial limits, e.g. sharp boundary limit. The vacuum should be thought of as coupling to smooth potentials.

Origin of Non-Universal Terms: 2007 - Kolomeisky and Straley

Non-universal terms come up only when there is a change in shape of a body under Casimir stresses.

A boundary should be thought of as an elastic membrane.

Geometry of an Elastic Membrane

Shape of a curved body given by:

- Total surface area
- Principal curvatures C_1 and C_2

Elastic stresses change these geometric properties of a membrane.

An elastic Hamiltonian will be an expansion in combinations of C_1 and C_2.

Elastic Hamiltonian for the Casimir Effect

The energy of a boundary is given by a local elastic Hamiltonian of the form:

$$H = \int ds \left(\gamma_0 + \gamma_{1a}(C_1 - C_2)^2 + \gamma_{1b} C_1 C_2\right)$$

where C_1 and C_2 are principal curvatures, and γ_{nm} are ω_0-dependent.
For a cylinder:

$$H = \int ds \left(\gamma_0 + \gamma_{1a} C_1^2(s)\right)$$

- Curvature stiffness coefficients have contributions from vacuum zero point energy!

Infinite Cylindrical Wedge: Setup

Perfectly conducting, no thermal fluctuations.

$$E_0 = \sum_n \frac{1}{2}\hbar\omega_n$$

ω_n are the allowed electromagnetic modes.

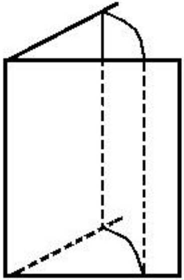


Figure: Cylindrical Wedge opening angle β, radius 'a'

Cylindrical Wedge: Allowed Modes

For a cylindrical wedge, allowed modes are:
Transverse Electric (TE) and Transverse Magnetic (TM) modes
under boundary conditions at the surface: $\mathbf{E}^{\parallel} = 0$ and $\mathbf{B}^{\perp} = 0$
Periodic boundary conditions at $\phi = 0$ and $\phi = \beta$

For brevity, we will discuss only TM modes.

TM Modes

Variable separable solutions to Maxwell's equations:

$$\sum_{m,n,\omega} e^{(\imath m z)} e^{(2\imath\pi n\phi/\beta)} e^{(-\imath\omega t)} R(\lambda\rho)$$

where $\lambda^2 = \frac{\omega^2}{c^2} - m^2$

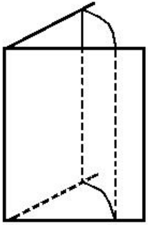


Figure: Wedge with angle β, radius 'a'

- Inside the wedge: $R(\rho\lambda) \propto J_{\frac{2\pi n}{\beta}}(\rho\lambda)$ Bessel functions of 1^{st} kind
- Outside the wedge: $R(\rho\lambda) \propto H_{\frac{2\pi n}{\beta}}(\rho\lambda)$ Hankel functions

TM Modes: Dirichlet Condition

Boundary conditions on TM modes $\Rightarrow R(a\lambda) = 0$

Have to add all the modes that satisfy this Condition.

Use the Argument Principle:

$$\sum_n z_n = \frac{1}{2\pi\imath} \oint_\Gamma z \frac{d}{dz} \ln(f(z))$$

where: z_n is a zero of $f(z)$ enclosed by the contour.

TM Casimir Energy

$$E \propto \frac{1}{2\pi\imath}\sum_n \int \frac{L}{2\pi} dm \oint \omega \frac{d}{d\omega} \ln\left(J_{\frac{2\pi n}{\beta}}(\lambda a) H_{\frac{2\pi n}{\beta}}(\lambda a)\right) d\omega$$

$\omega \to \imath\omega$, introducing polar coordinates, fixing units and factors, and inserting a cut off yields:

$$\frac{E}{L} = -\frac{\hbar c}{8\pi} \sum_{n=-\infty}^{\infty} \int_0^{\infty} \lambda^2 d\lambda \cdot F\left(\frac{\lambda c}{\omega_o}\right) \frac{d}{d\lambda} \ln\left[I_{\frac{2\pi n}{\beta}}(\lambda a) K_{\frac{2\pi n}{\beta}}(\lambda a)\right]$$

The $n = 0$ term (call it $\frac{E_a}{L}$) gives us a template for extracting universal and non-universal pieces.

TM Casimir Energy

$$E \propto \frac{1}{2\pi\imath}\sum_n \int \frac{L}{2\pi} dm \oint \omega \frac{d}{d\omega} \ln\left(J_{\frac{2\pi n}{\beta}}(\lambda a) H_{\frac{2\pi n}{\beta}}(\lambda a)\right) d\omega$$

$\omega \to \imath\omega$, introducing polar coordinates, fixing units and factors, and inserting a cut off yields:

$$\frac{E}{L} = -\frac{\hbar c}{8\pi} \sum_{n=-\infty}^{\infty} \int_0^{\infty} \lambda^2 d\lambda \cdot F\left(\frac{\lambda c}{\omega_o}\right) \frac{d}{d\lambda} \ln\left[I_{\frac{2\pi n}{\beta}}(\lambda a) K_{\frac{2\pi n}{\beta}}(\lambda a)\right]$$

The $n = 0$ term (call it $\frac{E_a}{L}$) gives us a template for extracting universal and non-universal pieces.

TM Casimir Energy

$$E \propto \frac{1}{2\pi\imath}\sum_n \int \frac{L}{2\pi} dm \oint \omega \frac{d}{d\omega} \ln\left(J_{\frac{2\pi n}{\beta}}(\lambda a) H_{\frac{2\pi n}{\beta}}(\lambda a)\right) d\omega$$

$\omega \rightarrow \imath\omega$, introducing polar coordinates, fixing units and factors, and inserting a cut off yields:

$$\frac{E}{L} = -\frac{\hbar c}{8\pi} \sum_{n=-\infty}^{\infty} \int_0^{\infty} \lambda^2 d\lambda \cdot F\left(\frac{\lambda c}{\omega_o}\right) \frac{d}{d\lambda} \ln\left[I_{\frac{2\pi n}{\beta}}(\lambda a) K_{\frac{2\pi n}{\beta}}(\lambda a)\right]$$

The $n = 0$ term (call it $\frac{E_a}{L}$) gives us a template for extracting universal and non-universal pieces.

Extracting Non-Universal Pieces

$$\frac{E_a}{L} = -\frac{\hbar c}{8\pi} \int_0^{\infty} \lambda^2 d\lambda \cdot F\left(\frac{\lambda c}{\omega_o}\right) \frac{d}{d\lambda} \ln\left[I_0\left(\lambda a\right) K_0\left(\lambda a\right)\right]$$

As $\lambda \to \infty$:

$$\ln\left[I_0(\lambda a) K_0(\lambda a)\right] \to \ln\left[\frac{1}{2\lambda a}\left(1 + \frac{1}{8a^2\lambda^2} + \mathbf{O}\left(\frac{1}{a^4\lambda^4}\right)\right)\right]$$
$$\sim \ln\left[\frac{1}{2a\lambda}\right] + \frac{1}{8a^2\lambda^2} + \mathbf{O}\left(\frac{1}{a^4\lambda^4}\right)$$

Extracting Non-Universal Pieces

$$\frac{E_a}{L} = -\frac{\hbar c}{8\pi} \int_0^{\infty} \lambda^2 d\lambda \cdot F\left(\frac{\lambda c}{\omega_o}\right) \frac{d}{d\lambda} \ln\left[I_0\left(\lambda a\right) K_0\left(\lambda a\right)\right]$$

As $\lambda \to \infty$:

$$\ln\left[I_0(\lambda a) K_0(\lambda a)\right] \to \ln\left[\frac{1}{2\lambda a}\left(1 + \frac{1}{8a^2\lambda^2} + \mathbf{O}\left(\frac{1}{a^4\lambda^4}\right)\right)\right]$$
$$\sim \ln\left[\frac{1}{2a\lambda}\right] + \frac{1}{8a^2\lambda^2} + \mathbf{O}\left(\frac{1}{a^4\lambda^4}\right)$$

Divergences in E_a: $\ln\left[\frac{1}{2a\lambda}\right]$

Put $\frac{d}{d\lambda}\ln\left[\frac{1}{2a\lambda}\right] = -\frac{1}{\lambda}$ as integrand of E_a:

$$\frac{E_a}{L} \to \frac{\hbar c}{8\pi}\int \lambda F\left(\frac{\lambda c}{\omega_0}\right) d\lambda \quad \text{geometry independent}$$

Hence, we subtract $\ln\left(\frac{1}{2a\lambda}\right)$ for $\frac{E_a}{L}$ without changing the force:

$$\frac{E_a}{L} \to -\frac{\hbar c}{8\pi}\int \lambda^2 d\lambda F\left(\frac{\lambda c}{\omega_0}\right)\frac{d}{d\lambda}\ln\left[2a\lambda I_0(a\lambda)K_0(a\lambda)\right]$$

Divergences in E_a: $\ln\left[\frac{1}{2a\lambda}\right]$

Put $\frac{d}{d\lambda}\ln\left[\frac{1}{2a\lambda}\right] = -\frac{1}{\lambda}$ as integrand of E_a:

$$\frac{E_a}{L} \to \frac{\hbar c}{8\pi}\int \lambda F\left(\frac{\lambda c}{\omega_0}\right) d\lambda \quad \text{geometry independent}$$

Hence, we subtract $\ln\left(\frac{1}{2a\lambda}\right)$ for $\frac{E_a}{L}$ without changing the force:

$$\frac{E_a}{L} \to -\frac{\hbar c}{8\pi}\int \lambda^2 d\lambda F\left(\frac{\lambda c}{\omega_0}\right)\frac{d}{d\lambda}\ln\left[2a\lambda I_0(a\lambda)K_0(a\lambda)\right]$$

Divergences in E_a: $\frac{1}{8a^2\lambda^2}$

Next order divergence is $\dfrac{1}{8a^2(1+\lambda^2)} \sim \ln\left(\dfrac{a\omega_0}{c}\right)$

$$\frac{E_a}{L} = -\frac{\hbar c}{8\pi a^2}\int \lambda^2 d\lambda \cdot \frac{d}{d\lambda}\left(\ln\left[2\lambda I_0(\lambda)K_0(\lambda)\right] - \frac{1}{8(1+\lambda^2)}\right)$$
$$-\frac{\hbar c}{8\pi a^2}\int \lambda^2 d\lambda \cdot F\left(\frac{c\lambda}{a\omega_0}\right)\frac{d}{d\lambda}\frac{1}{8(1+\lambda^2)}$$

Divergences in E_a: $\frac{1}{8a^2\lambda^2}$

Next order divergence is $\dfrac{1}{8a^2(1+\lambda^2)} \sim \ln\left(\dfrac{a\omega_0}{c}\right)$

$$\frac{E_a}{L} = -\frac{\hbar c}{8\pi a^2}\int \lambda^2 d\lambda \cdot \frac{d}{d\lambda}\left(\ln\left[2\lambda I_0(\lambda)K_0(\lambda)\right] - \frac{1}{8(1+\lambda^2)}\right)$$
$$-\frac{\hbar c}{8\pi a^2}\int \lambda^2 d\lambda \cdot F\left(\frac{c\lambda}{a\omega_0}\right)\frac{d}{d\lambda}\frac{1}{8(1+\lambda^2)}$$

Final Energy Expression for TM Modes

$$\frac{E_0}{L} = \alpha s - \frac{\xi s}{a^2} + \frac{\hbar c}{8\pi a^2}\left(0.28614 + \frac{2\zeta(3)}{\beta^2} + \frac{1}{4}\ln\left(\frac{2}{\beta}\right)\right)$$

where $\alpha \sim \hbar\omega_0 \frac{\omega_0^2}{c^2}\int \lambda^2 F(\lambda)d\lambda$ and $\xi \sim \hbar\omega_0 \int F(\lambda)d\lambda$

Integral form of non-universal terms for arbitrary cross section:

$$\frac{E_0(\omega_0)}{L} = \int ds(\alpha - \xi C_1^2(s))$$

Final Energy Expression for TM Modes

$$\frac{E_0}{L} = \alpha s - \frac{\xi s}{a^2} + \frac{\hbar c}{8\pi a^2}\left(0.28614 + \frac{2\zeta(3)}{\beta^2} + \frac{1}{4}\ln\left(\frac{2}{\beta}\right)\right)$$

where $\alpha \sim \hbar\omega_0 \frac{\omega_0^2}{c^2}\int \lambda^2 F(\lambda)d\lambda$ and $\xi \sim \hbar\omega_0 \int F(\lambda)d\lambda$

Integral form of non-universal terms for arbitrary cross section:

$$\frac{E_0(\omega_0)}{L} = \int ds(\alpha - \xi C_1^2(s))$$

Conclusions

- Cut off function has a physically understandable basis
- The origin of non-universal terms in Casimir self energy calculations can be explained by viewing the surface as an elastic membrane
- The case of an infinite cylindrical wedge is an explicit example of the elasticity theory proposal

Acknowledgements

- Eugene Kolomeisky (University of Virginia)
- Luke Langsjoen (University of Virginia)
- Joe Straley (University of Kentucky)