THE EFFECT OF THERMAL FLUCTUATIONS ON THE PROBLEM OF EULER BUCKLING INSTABILITY

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Introduction

- The Euler buckling instability is a property of long thin rods at zero temperature.
- If you compress such a rod, the unbent configuration will remain stable until a critical force F_c is surpassed, when the rod will suddenly start to buckle.
- I will investigate how this phenomenon is affected by thermal fluctuations in the limit of high temperature.

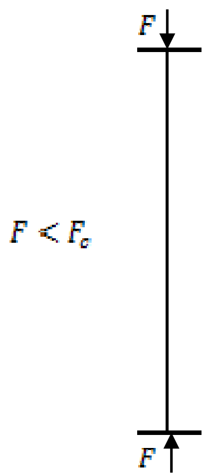
Motivation

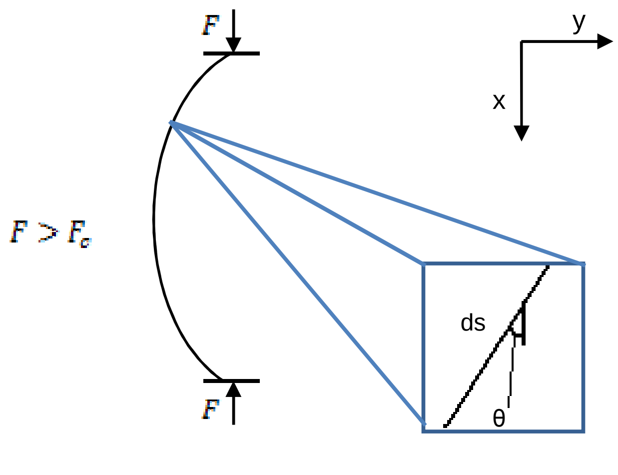
- It's physically interesting and uses cool mathematics.
- Applicable to microtubules, intermediate filaments and actin filaments in the cytoskeleton. [1,2,3,4]
- Carbon nanotubes.[5,6]

[1] C.P. Brangwynne et al, *J.Cell. Biol.*, 2006, **173**(5), 733.
[2] M. Kurachi et al, *Cell Motility and the Cytoskeleton*, 1995, **30**(3), 221.
[3] O. Chaudhuri et al, *Nature*, 2005, **445**, 295
[4] K.D. Costa et al, *Cell Motility and the Cytoskeleton*, 2002, **52**(4), 266
[5] M.R. Falvo et al, Nature, 1997, **389**, 582
[6] H.S. Yap Nano Lett., 2007, **7**(5), 1149

Introduction to the Euler Buckling Instability

Unclamped case:





$$A\frac{d\theta}{ds} = -Fy \quad \Rightarrow \quad A\frac{d^2\theta}{ds^2} = -F\sin\theta \quad \Rightarrow \quad H = \int_0^L ds\left[\frac{A}{2}\left(\frac{d\theta}{ds}\right)^2 + F\cos\theta\right]$$

$$\boxed{A = YI}$$

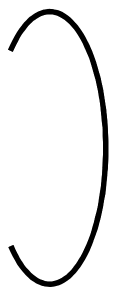
Expand θ in a Fourier series:

$$\theta = \sum_{n=1}^{\infty} \theta_n \cos\left(\frac{n\pi s}{L}\right)$$

$$\theta^1 = \theta_1 \cos\left(\frac{\pi s}{L}\right)$$

$\theta_1 \ll 1$

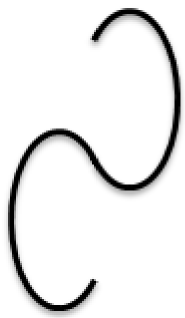
$\theta_1 > 1$



$$\theta^2 = \theta_2 \cos\left(\frac{2\pi s}{L}\right)$$

$\theta_2 \ll 1$

$\theta_2 > 1$



$$H = \int_0^L ds \left[\frac{A}{2} \left(\frac{d\theta}{ds} \right)^2 + F \cos\theta \right]$$

Expand θ in a Fourier series:

$$\theta = \sum_{n=1}^{\infty} \theta_n \cos\left(\frac{n\pi s}{L}\right)$$

Approximate:

$$\cos\theta \cong \mathbf{1} - \frac{\theta^2}{2!}$$

$$H = \sum_{n,m=1}^{\infty} \int ds \left[\frac{A\pi^2}{2L^2} nm\theta_n\theta_m \sin\left(\frac{n\pi s}{L}\right) \sin\left(\frac{m\pi s}{L}\right) - \frac{F}{2}\theta_n\theta_m \cos\left(\frac{n\pi s}{L}\right) \cos\left(\frac{m\pi s}{L}\right) \right]$$

$$= \sum_{n=1}^{\infty} \theta_n{}^2 \left(A\frac{\pi^2 n^2}{4L} - \frac{FL}{4} \right)$$

Euler instability! When this becomes negative we will have buckling.

$$\boxed{F_{c,n} = A\frac{\pi^2 n^2}{L^2}}$$

Now, include temperature

How do thermal fluctuations affect this phenomenon?

Range of applicability

- Concrete pillars and structural beams clearly have a negligible influence from thermal fluctuations. (A >> LT)

- Biological filaments and nanodevices, however, can experience large non-trivial effects from thermal motion. (A ~ LT)

The Persistence Length

- $\frac{TL}{A} \equiv \frac{L}{L_p}$ where $L_p = \frac{A}{T}$ is the persistence length

- L_p is a rough measure of the maximum distance at which different parts of the rod are 'aware' of each other. It is also known as a correlation length.

$$H_0 = \frac{A}{2}\int_0^L ds \left(\frac{d\theta}{ds}\right)^2 \sim AL\frac{\theta^2}{L^2} \sim T \quad \Rightarrow \quad \langle\theta^2\rangle \sim \frac{TL}{A} \sim \frac{L}{L_p}$$

Averaging of Thermal Fluctuations

- Our probability distribution is $P = e^{-\frac{H(\theta)}{T}}$

- Write $\theta = \theta^1 + \tilde{\theta} = \theta_1 \cos\left(\frac{\pi s}{L}\right) + \sum_{n=2}^{\infty} \theta_n \cos\left(\frac{n\pi s}{L}\right)$

- Perform a renormalization group calculation on $\tilde{\theta}$. i.e., average over $\tilde{\theta}$ to obtain a probability distribution for θ_1 :

$$\bar{P} = e^{-\frac{\bar{H}(\theta^1)}{T}} = \prod_{n=2}^{\infty} \int d\theta_n e^{-\frac{H(\theta^1,\tilde{\theta})}{T}}$$

- Write $H = H_0(\theta_1) + H_0(\tilde{\theta}) + E_p$

where $H_0(\theta_1) = \frac{A\pi^2}{4L}\theta_1{}^2$, $H_0(\tilde{\theta}) = \sum_{n=2}^{\infty} \frac{A\pi^2 n^2}{4L}\theta_n{}^2$

and $E_p = F\int_0^L ds \cos(\theta^1 + \tilde{\theta})$

$$\bar{P} = e^{-\frac{\bar{H}(\theta^1)}{T}} = e^{-\frac{H_0(\theta^1)}{T}} \prod_{n=2}^{\infty} \int d\theta_n e^{-\frac{E_p}{T}} e^{-\frac{H_0(\tilde{\theta})}{T}}$$

Expand this term

Average over this distribution

- Taking the logarithm of $\bar{P}$ and expanding both the exponential and the logarithm for large T, we obtain:

$$\bar{H}(\theta_1) = H_0(\theta_1) + \langle E_p \rangle_o - \frac{1}{2T}\left[\langle {E_p}^2 \rangle_o - {\langle E_p \rangle_o}^2\right] + \cdots$$

- This is known as a cumulant expansion.

$$\langle E_p \rangle_o = F \int_0^L ds \langle \cos(\theta^1 + \tilde{\theta}) \rangle_o$$

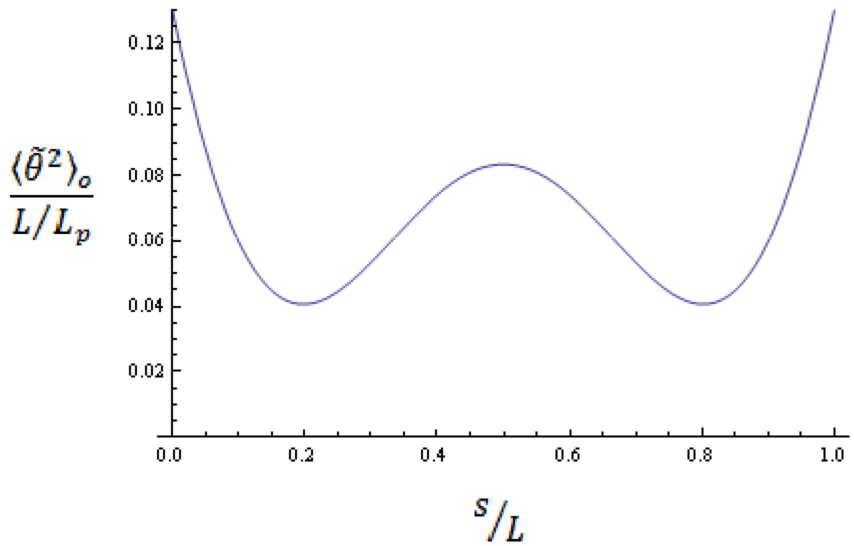
$$= F \int_0^L ds \cos\theta^1 \langle \cos\tilde{\theta} \rangle_o$$

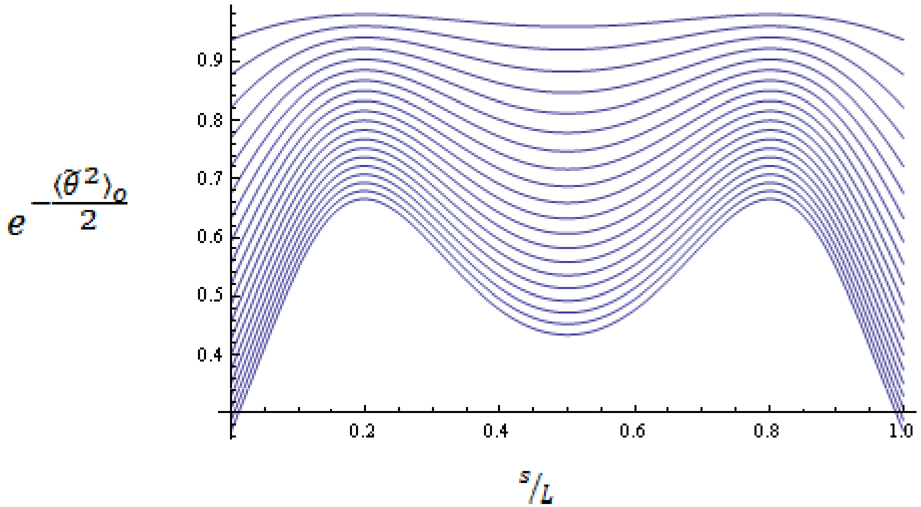
$$= F \int_0^L ds \cos\theta^1 \, e^{-\frac{\langle \tilde{\theta}^2 \rangle_o}{2}}$$

$$\cong F \int_0^L ds \left(1 - \frac{{\theta_1}^2}{2} \cos^2\left(\frac{\pi s}{\mathrm{L}}\right)\right) e^{-\frac{\langle \tilde{\theta}^2 \rangle_o}{2}}$$

$$\langle \tilde{\theta}^2 \rangle_o = \frac{2TL}{A\pi^2} \left(\frac{\pi^2}{24} - \cos^2\left(\frac{\pi s}{L}\right) + \frac{\pi^2}{2} \left(\frac{s}{L} - \frac{1}{2} \right)^2 \right)$$

$$\langle\tilde{\theta}^2\rangle_o = \frac{2TL}{A\pi^2}\left(\frac{\pi^2}{24} - \cos^2\left(\frac{\pi s}{L}\right) + \frac{\pi^2}{2}\left(\frac{s}{L} - \frac{1}{2}\right)^2\right)$$





Extrema occur when:

$$\sin\left(\frac{2\pi s}{L}\right) = \frac{\pi}{2} - \frac{\pi s}{L}$$

- Using the method of steepest descent for large T, we evaluate the integral and find:

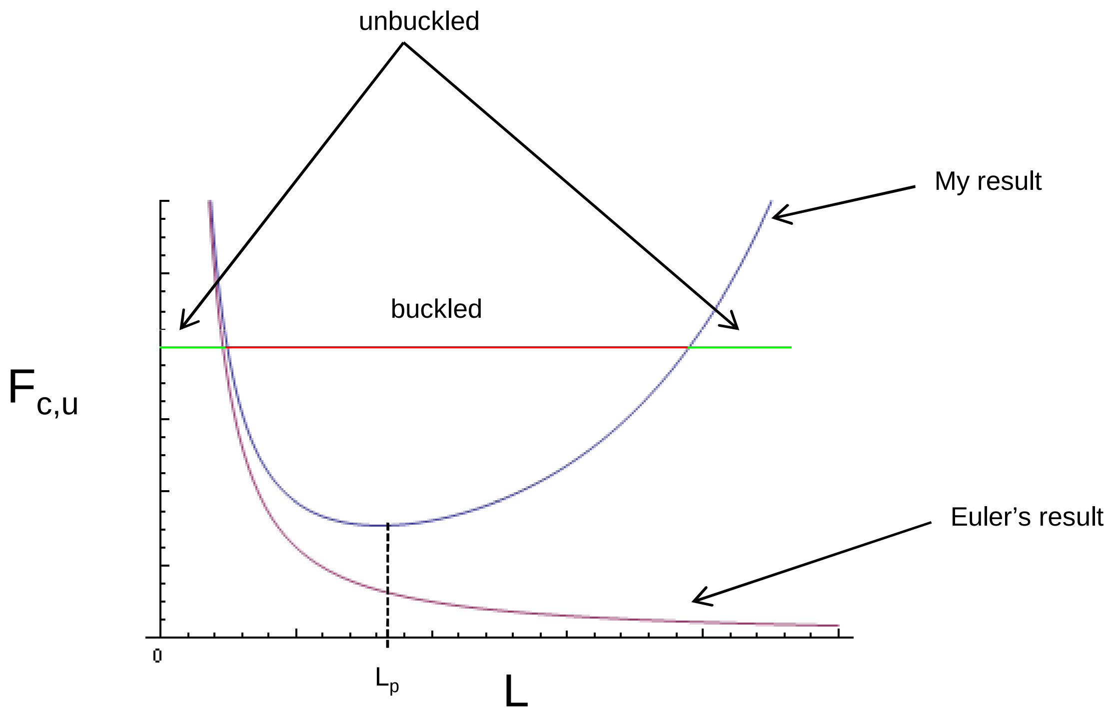
Including thermal fluctuations:

$$F_{c,u} \cong 0.955 e^{2.03\frac{TL}{A}} \sqrt{\frac{AT}{L^3}}$$

Disregarding thermal fluctuations (zero temperature):

$$F_{c,0} = \frac{A\pi^2}{L^2}$$

For finite T, there is a minimum critical force!



- Interestingly, this method works reasonably well even for small T!
- If we expand the exponential below to first order in T:

$$\langle E_p \rangle_o \cong \int_0^L ds\, F \left(1 - \theta_1{}^2 \cos^2\left(\frac{\pi s}{L}\right)\right) e^{-\frac{\langle \tilde{\theta}^2 \rangle_o}{2}}$$

we obtain

$$F_c \cong F_{c,0}\left(\mathbf{1} + .0327\frac{TL}{A}\right)$$

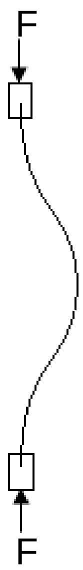
- This is quite close to the result of Baczynski et al[1], who solved this problem in the limit of small T:

$$F_c \cong F_{c,0}\left(\mathbf{1} + .0380\frac{TL}{A}\right)$$

- This indicates that our method captures the essential physics of the entire temperature range.

[1] K. Baczynski, R. Lipowsky, and J. Kierfeld. Phys. Rev. E, 2007, **76**, 061914

The case of clamped ends:



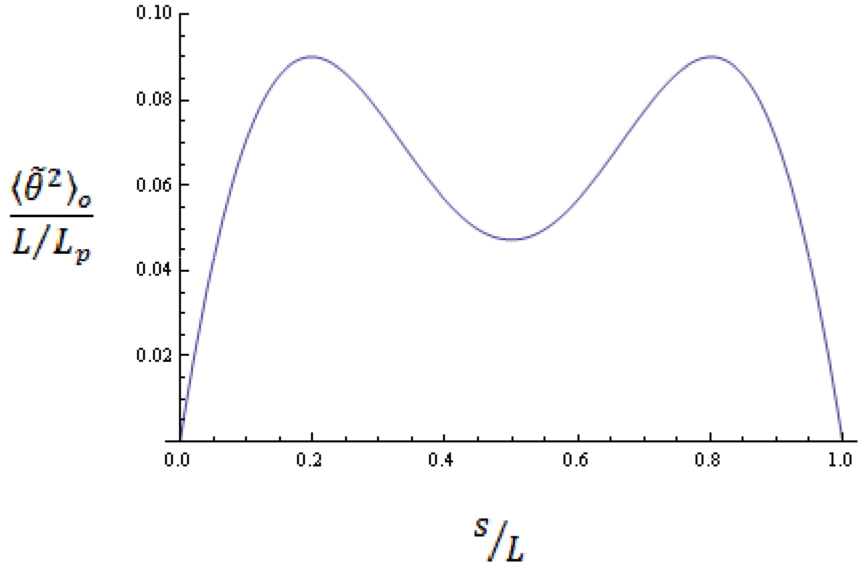
Now, $\theta = \sum_{n=1}^{\infty} \theta_n \sin\left(\frac{n\pi s}{L}\right)$

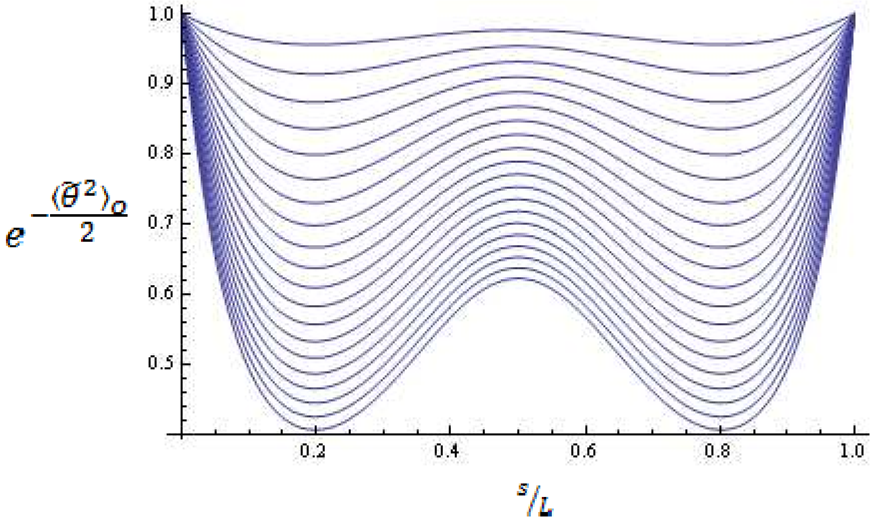
The calculation proceeds very similarly, except that now:

$$\langle E_p \rangle_o \cong \int_0^L ds\, F\left(1 - \sin^2\left(\frac{\pi s}{L}\right)\right) e^{-\frac{\langle \tilde{\theta}^2 \rangle_o}{2}}$$

$$\langle \tilde{\theta}^2 \rangle_o = \frac{2TL}{A\pi^2}\left(\frac{\pi^2}{8} - \sin^2\left(\frac{\pi s}{L}\right) - \frac{\pi^2}{2}\left(\frac{s}{L} - \frac{1}{2}\right)^2\right)$$

$$\langle\tilde{\theta}^2\rangle_o = \frac{2TL}{A\pi^2}\left(\frac{\pi^2}{8} - \sin^2\left(\frac{\pi s}{L}\right) - \frac{\pi^2}{2}\left(\frac{s}{L} - \frac{1}{2}\right)^2\right)$$





Now we perform a steepest descent calculation around s = 0 and s = L.

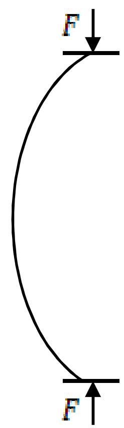
Interestingly, we get a very different answer!

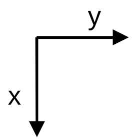
$$F_{c,cl} \cong \frac{LT^3}{64A^2} \qquad \text{compared with:} \qquad F_{c,u} \cong 0.955e^{2.03\frac{TL}{A}}\sqrt{\frac{AT}{L^3}}$$

Why the difference?

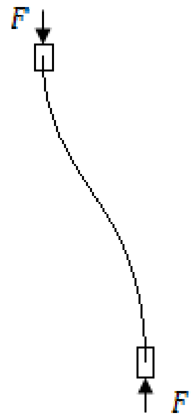
- In the region close to critical buckling, we may approximate $\theta_1 << 1$. Here the rod is approximately shaped like a sine wave.
- In this limit, the unclamped case corresponds to negligible overall lateral (y) displacement of the rod, while the clamped case will have a lateral displacement proportional to θ_1 :

$$\theta^1 = \theta_1 \cos\left(\frac{\pi s}{L}\right)$$





$$\theta^1 = \theta_1 \sin\left(\frac{\pi s}{L}\right)$$



- This indicates that the clamped and unclamped cases are physically quite different from one another.

Conclusions

- In the limit of high temperature it has been shown that F_c increases with both length and temperature.
- Because F_c decreases with length for small lengths, F_c is a non monotonic function of L and it will have a nonzero minimum value.

Acknowledgements

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