Entanglement, chaos and order

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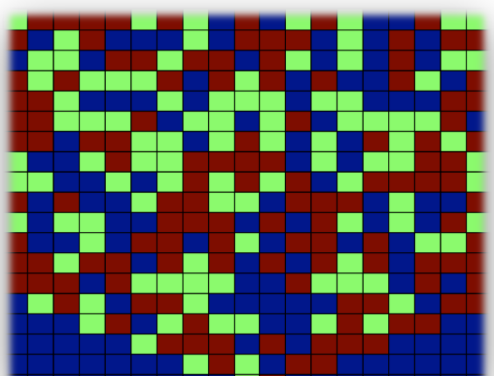
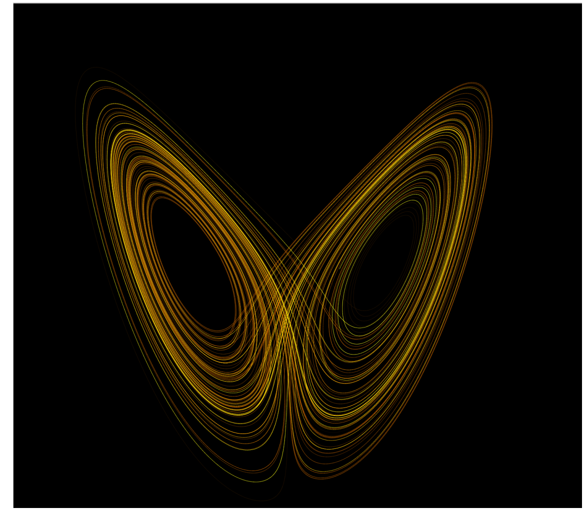
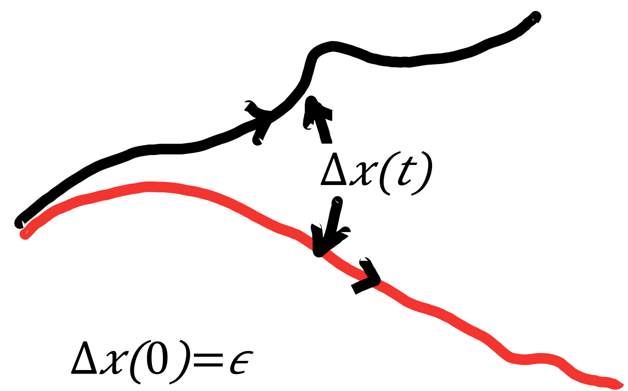


Univ. of Virginia, Nov 30th, 2017

Chaos: the butterfly effect



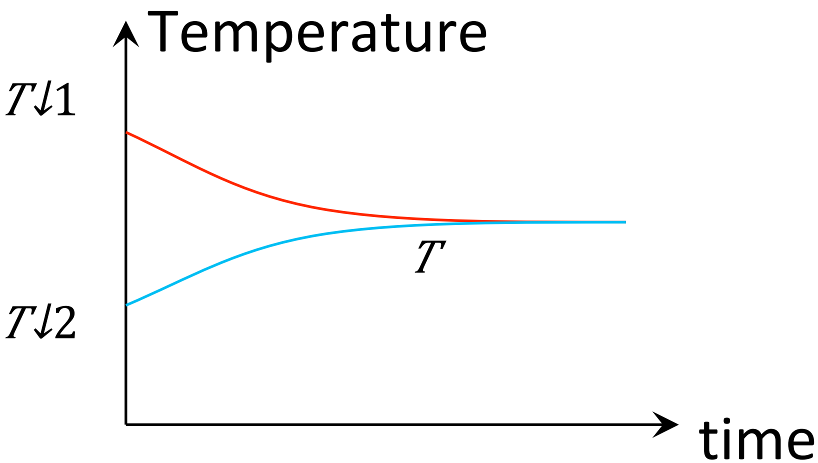


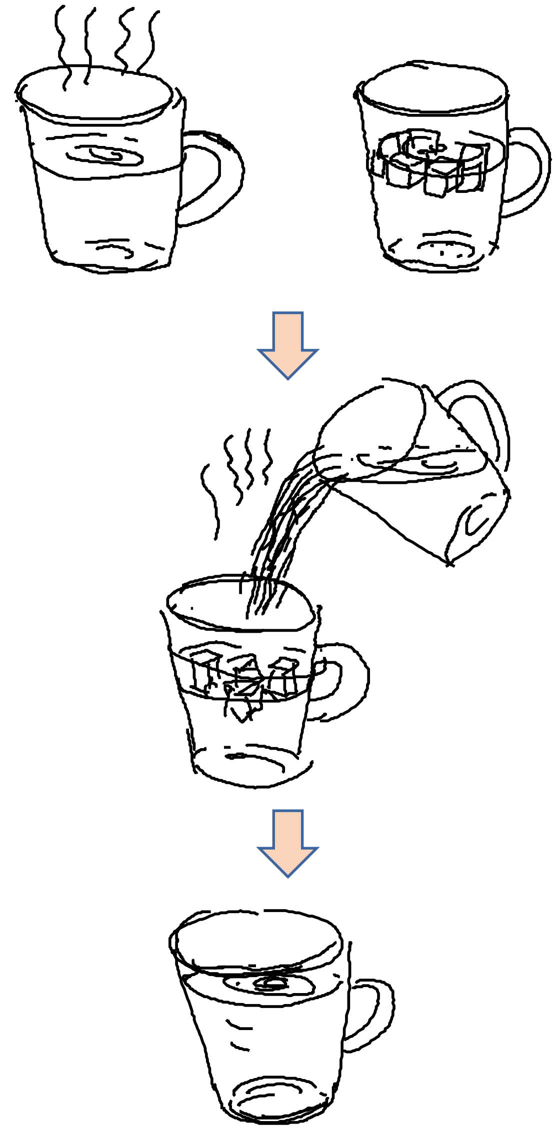
$\Delta x(0) = \epsilon$

$\Delta x(t) \sim e^{\lambda t} \epsilon$

Chaos and thermalization

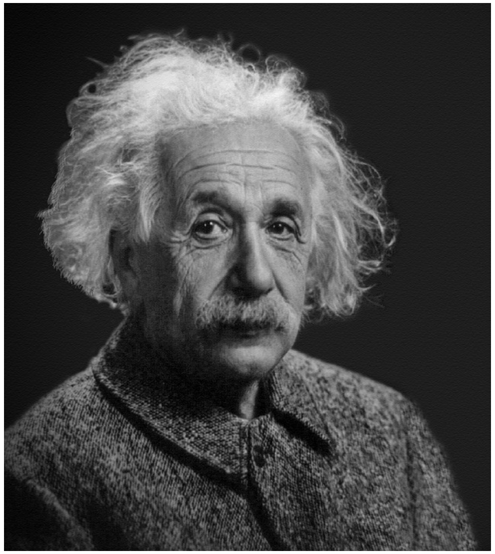
- Chaos ➔ ignorance
- A lot of chaos ➔ new knowledge!





- $x_i(t)[x_j(0), p_j(0)]$ very complicated. Looks random
- Many-body chaos ➔ thermalization
- Thermodynamics emerges from ignorance.

Chaos and thermalization



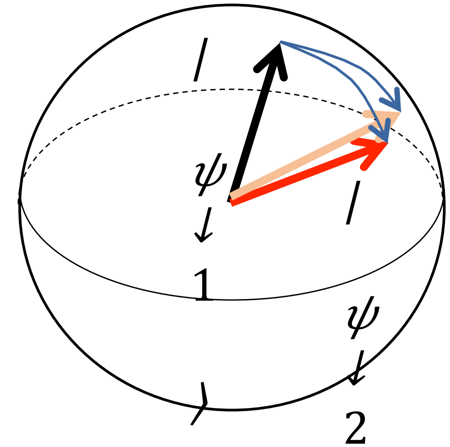
It (classical thermodynamics) is the only physical theory of universal content which I am convinced will never be overthrown, within the framework of applicability of its basic concepts.

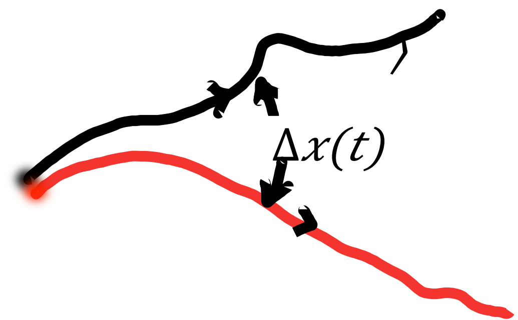
-- Albert Einstein

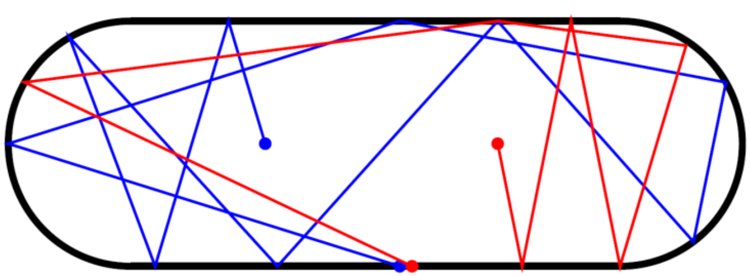
- How about quantum mechanics?

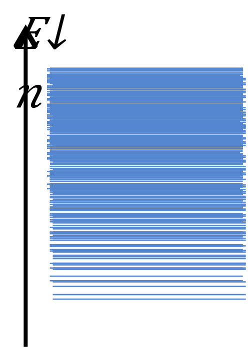
Quantum chaos

- $i\partial/\partial t\,|\psi\rangle = H|\psi\rangle$
- No actual chaos if Hilbert space dimension D is finite
- Initial condition $\Delta x(0)$ is blurred by the uncertainty principle
- Chaos can be defined in limit $D \to \infty$



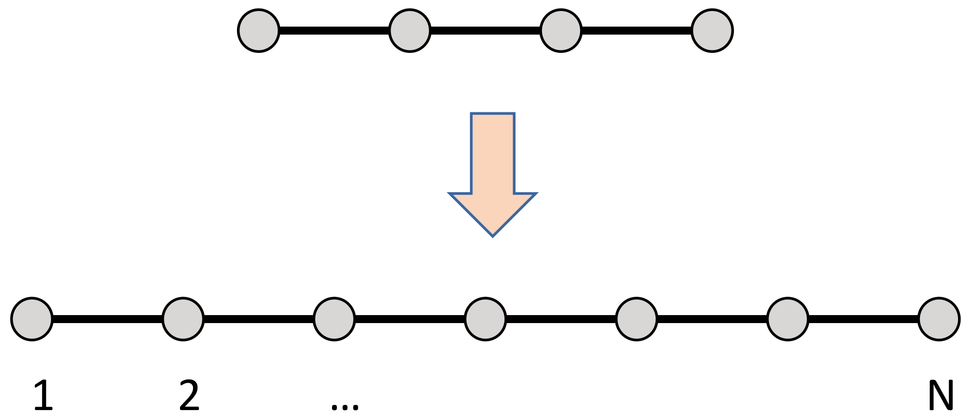






Chaos in high energy limit

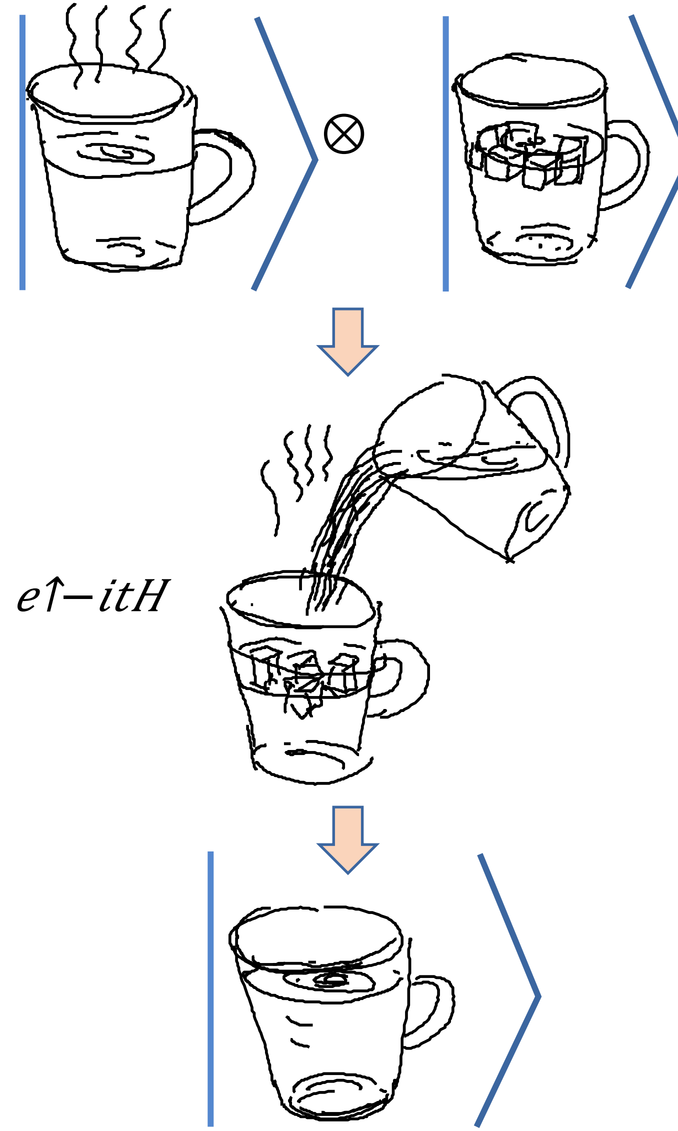
Many-body quantum chaos



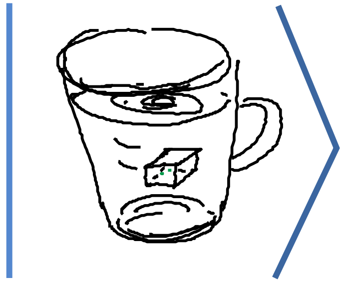
- Many-body system
- Hilbert space dimension increases exponentially
- $D=2^N$ for spin chain
- State with a finite energy density $E=N\epsilon$ has diverging density of state at large N. $\rho(E)\propto e^{Ns(E)}$
- Quantum chaos is generic in the thermodynamic limit $N\to\infty$.
- Chaos ➔ ignorance
- A lot of chaos ➔ new knowledge!

Many-body quantum chaos

- Will thermalization be different for an isolated quantum system?
- Quantum system is described by a density matrix ρ
- $\rho=\sum_n p_n |n\rangle\langle n|$
- Von Neumann entropy $S=-tr(\rho\log\rho)=-\sum_n p_n \log p_n$
- A glass of "pure state water" has no entanglement entropy
- Will it taste different?
- No, unless you are "exponentionally sensitive"



Thermalization from entanglement

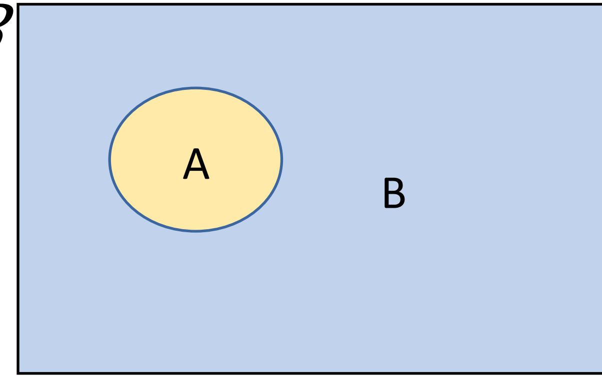


- There is no way to **locally** distinguish pure state water from either pure state water with different initial states, or mixed state water.
- Orthogonal states $|\psi_1\rangle, |\psi_2\rangle$ evolves to orthogonal states $|\psi_1(t)\rangle, |\psi_2(t)\rangle$, but the local reduced density matrices $\rho_1(t) \simeq \rho_2(t)$ are almost the same.
- Thermal entropy emerges from entanglement entropy.

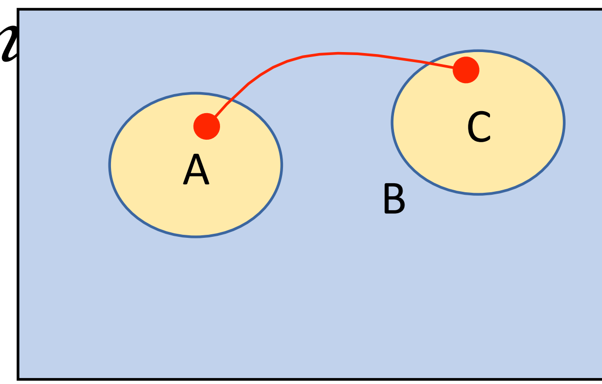
Deutsch '91, Srednicky '94

Entanglement entropy

- $|\psi\rangle=\sum_{n} \sqrt{p_n}\, |n\rangle_A \otimes |n\rangle_B$



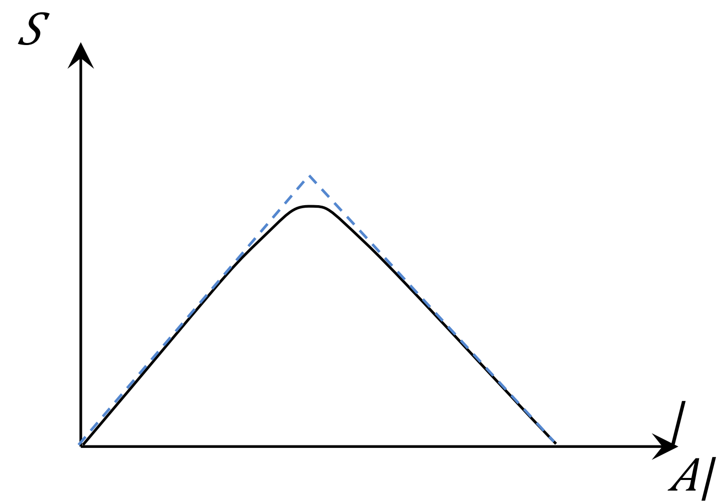
- Region A is in a mixed state.
  State $|n\rangle_A$ has probability p_n
- A is entangled with its complement B.
- $S_A = S_B = -\sum_{n} p_n \log p_n$

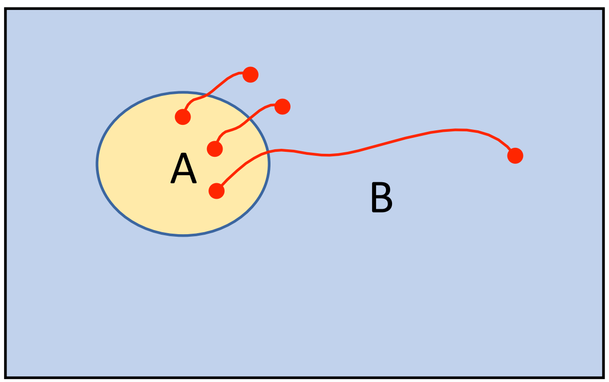


- Mutual information:
- $I(A:C)=S(A)+S(C)-S(AC)$
- Measure of correlation between A and C.

Entanglement in thermal state

- Thermalization ➔ Volume law entropy for small subsystems
- Entanglement is not in simple EPR pair form.
- Thermalization from onlocal, multipartite entanglement.





Entanglement from EPR pairs

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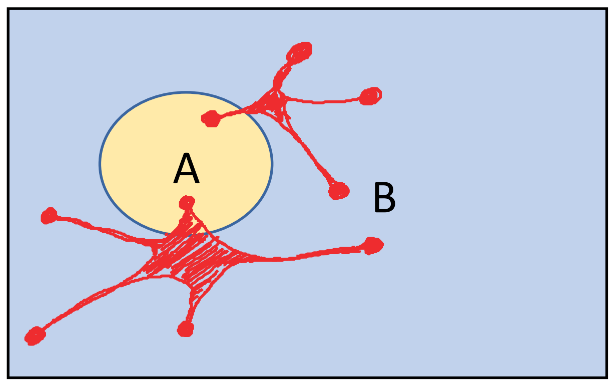
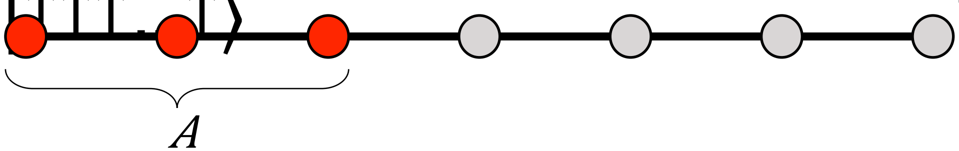


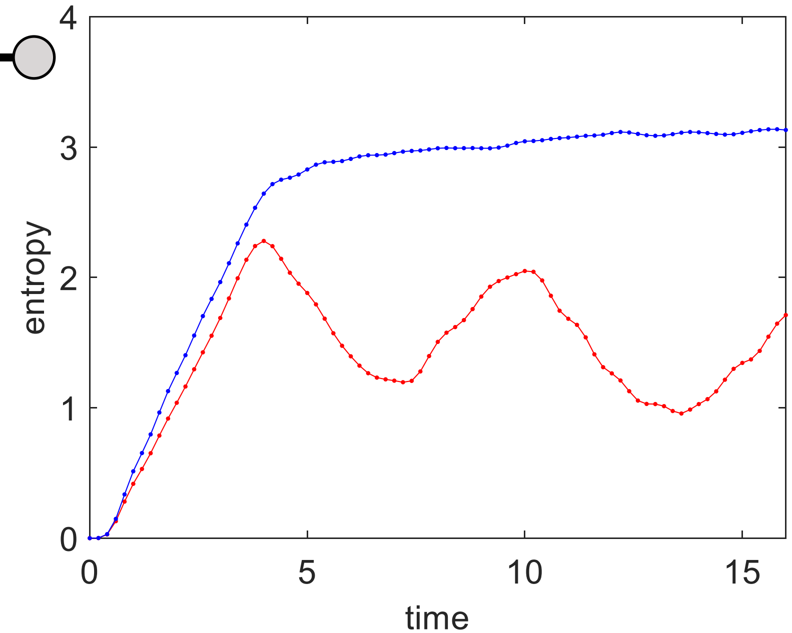
Multipartite entanglement

√

Thermalization after a quench

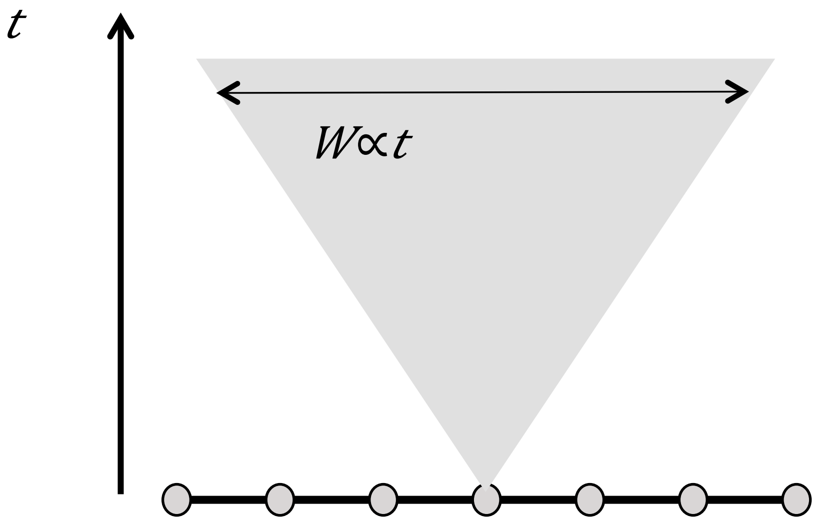
- Example: 1+1D Ising model
- $H = -J\sum_n \sigma_n^z \sigma_{n+1}^z - h_x \sum_n \sigma_n^x - h_z \sum_n \sigma_n^z$
- $h_x = 0$ integrable (equilvalent to free fermions).
- Time evolution starts from a product state, such as $|\uparrow\uparrow\uparrow\ldots\uparrow\rangle$



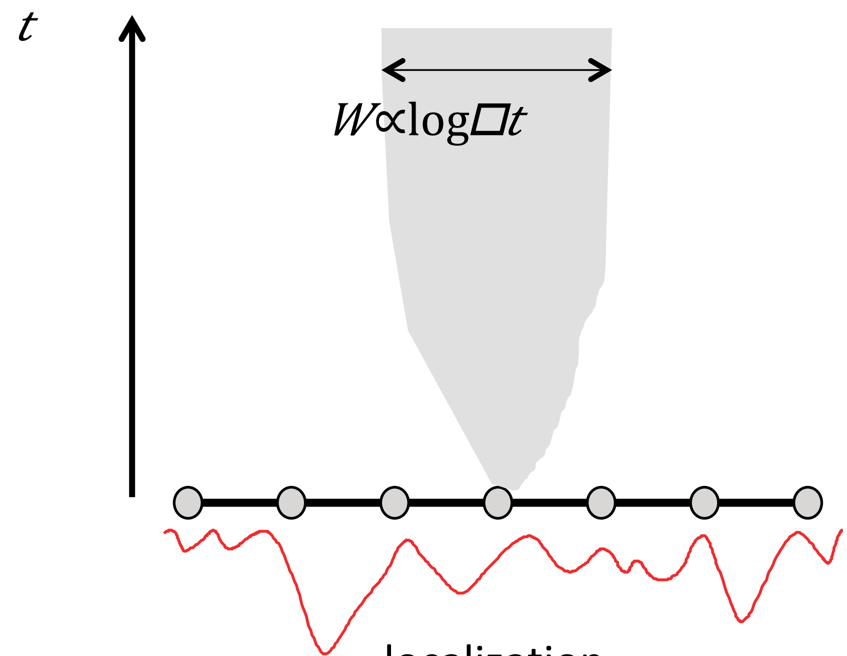


- Thermalization $S(t) \propto t$ till saturation
- Absence of thermalization: exact solvable model, or many-body localization.

Thermalization vs localization



thermalization

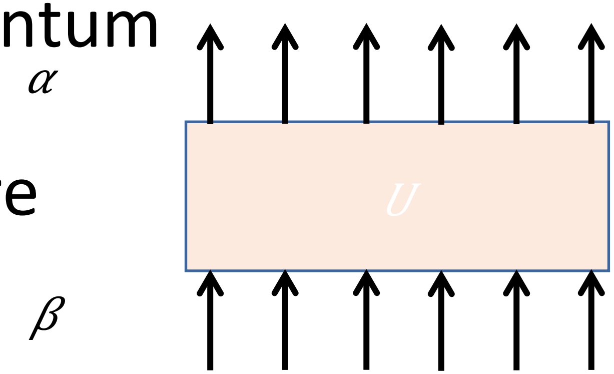


localization

- Entanglement growth = losing local information
- Localization = local information stays local (therefore slower entanglement growth)

Calabrese & Cardy '05
Amico et al RMP '08,
Bardarson, Pollmann, Moore '12

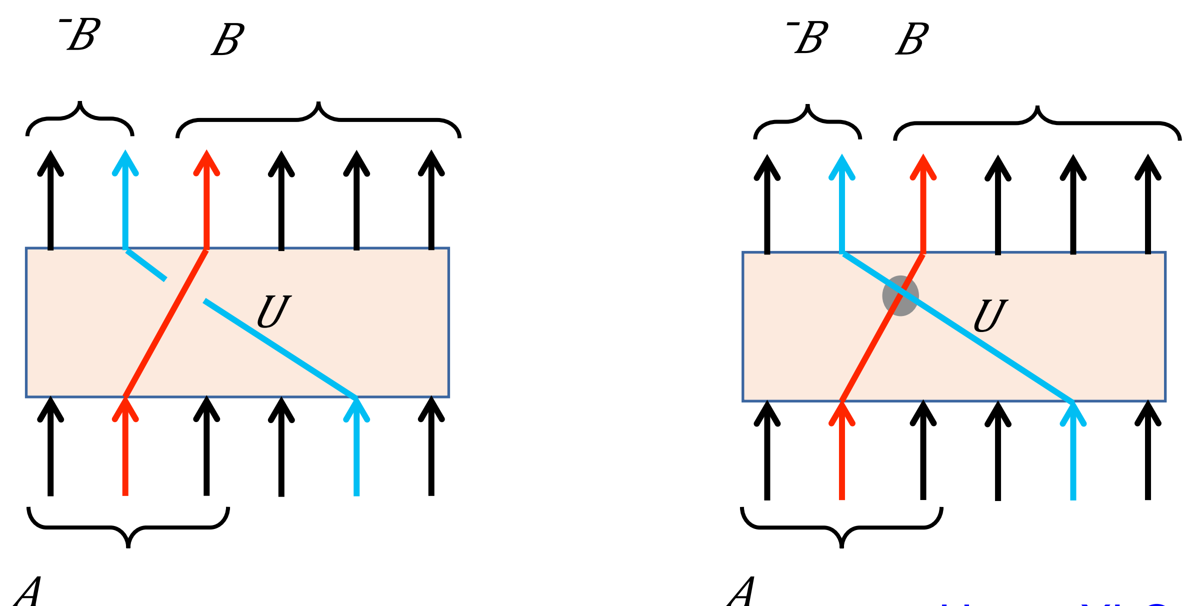
Entanglement measure of chaos



- Chaos = nonlocal spreading of quantum information
- Wanted: an entanglement measure of information spreading
- Trick: Convert the unitary operator e^{-itH} into a state in a bigger system
- Measure correlation by mutual information
- $e^{-itH} = U_{\alpha\beta}\,|\alpha\rangle\langle\beta| \to |\Psi\rangle = \frac{1}{\sqrt{D}}\,U_{\alpha\beta}\,|\alpha\rangle|\beta\rangle$
- Example: $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right)$
  $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\uparrow\rangle - |\downarrow\rangle|\downarrow\rangle\right)$

Entanglement measure of chaos

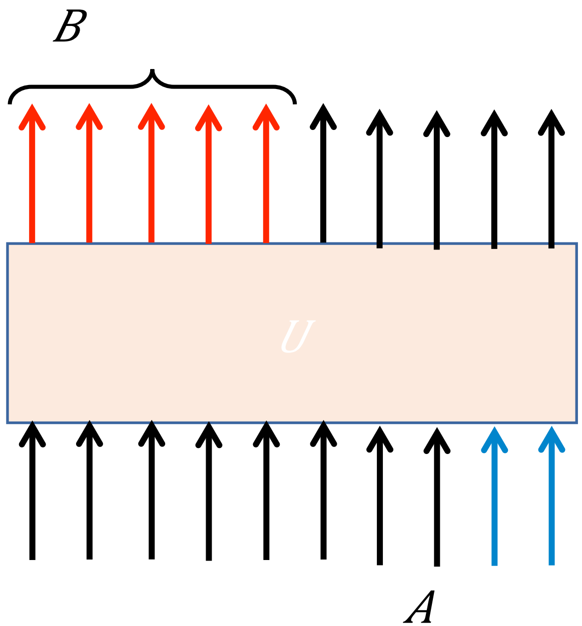
- Unitary evolution ➔ Maximal entanglement
- Correlation ➔ mutual information
- Operator scrambling ➔ suppression of mutual information
- Chaos ➔ $I(A{:}B)+I(A{:}\bar{B})\ll I(A{:}B\bar{B})$

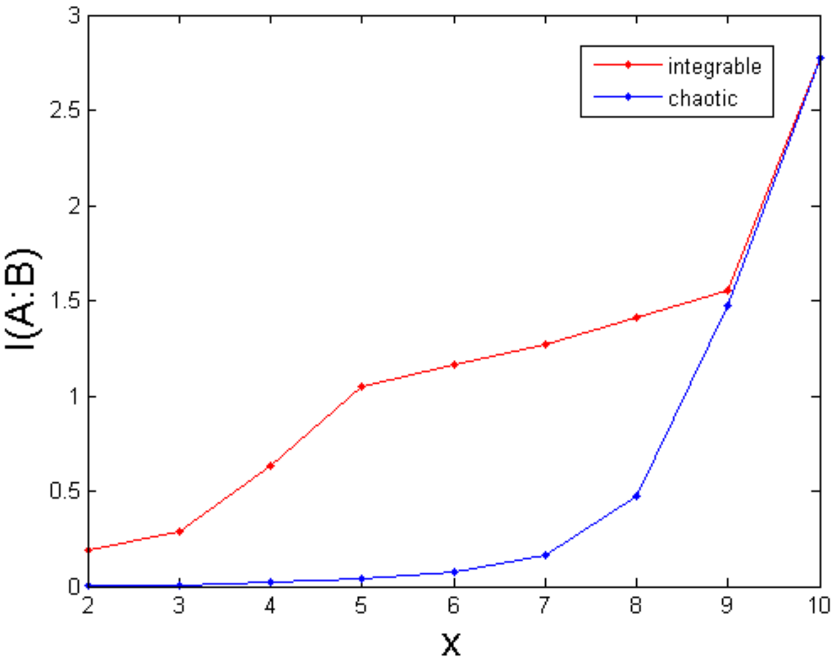


Hosur XLQ Roberts Yoshida '15

Entanglement measure of chaos

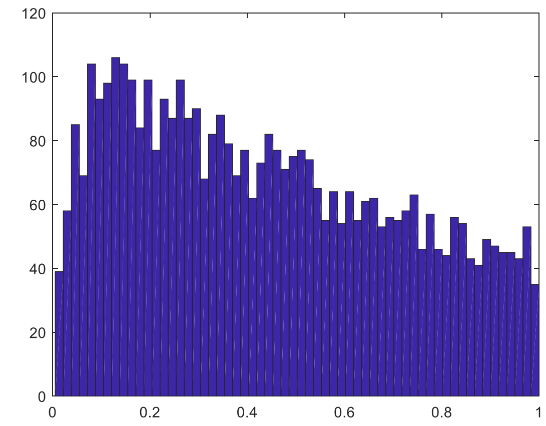
- Ising model numerics
- Chaos ⇒I(A:B) small as long as $A+B<\mathrm{L}$ (system size)





How to get more analytic results?

- “Solvable” chaotic models:
- Random matrix theory
  Level statistics of a chaotic Hamiltonian agrees well with that of a random matrix.

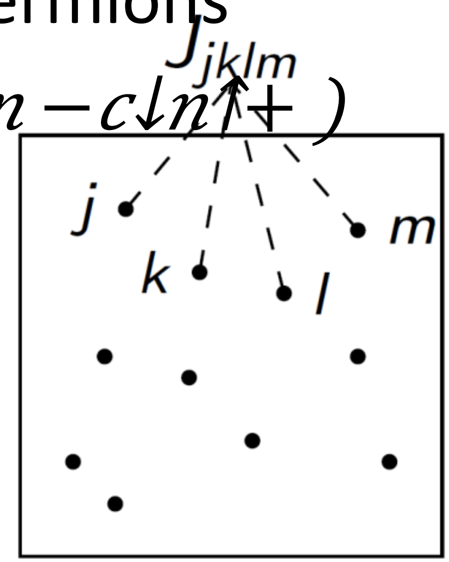


- Holographic duality
  Some strongly coupled quantum field theories are dual to weakly coupled gravity.

- Sachdev-Ye-Kitaev model and generalizations

Sachdev-Ye-Kitaev (SYK) model

- Random nonlocal interaction for Majorana fermions $H=\sum_{ijkl} J_{ijkl}\,\chi_i\,\chi_j\,\chi_k\,\chi_l$ with independent coupling $\overline{J_{ijkl}^2} = N^{-3}/2\, J^2$.
- $\{\chi_i,\chi_j\}=2\delta_{ij}$
- N Majorana fermions = $N/2$ complex fermions
- $\chi_{2n-1}=c_n+c_n^+$, $\chi_{2n}=-i(c_n-c_n^+)$ (Bogoliubov quasiparticles)



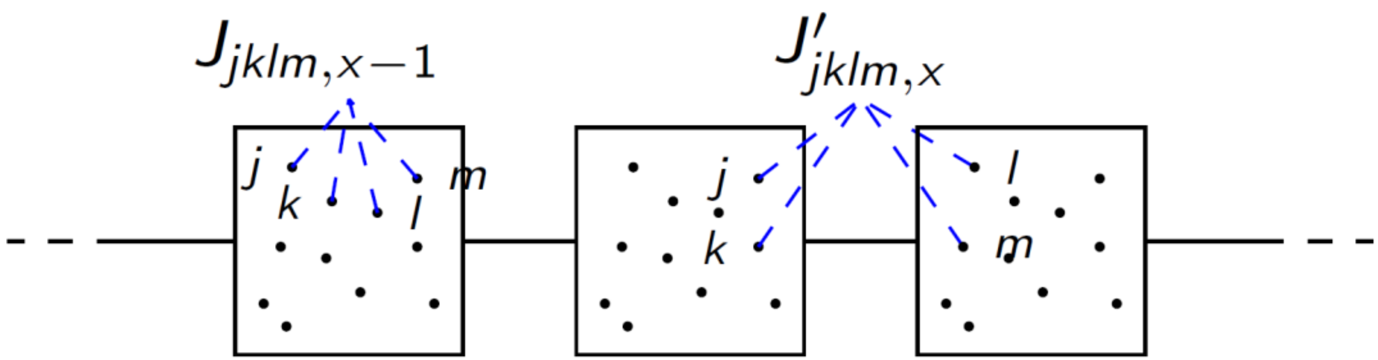
Sachdev, Ye, 1993; Kitaev 2015; Maldacena, Stanford 2016

Generalized SYK model

- Couple SYK sites by random coupling.
- For example in 1d,

$$H=\sum_{x=1}^{M}\left[\sum_{j<k<l<m}\underbrace{J_{jklm,x}\ \chi_{j,x}\chi_{k,x}\chi_{l,x}\chi_{m,x}}_{\text{SYK term}}+\sum_{j<k;l<m}\underbrace{J'_{jklm,x}\ \chi_{j,x}\chi_{k,x}\chi_{l,x+1}\chi_{m,x+1}}_{\text{Nearest neighbour coupling}}\right]$$

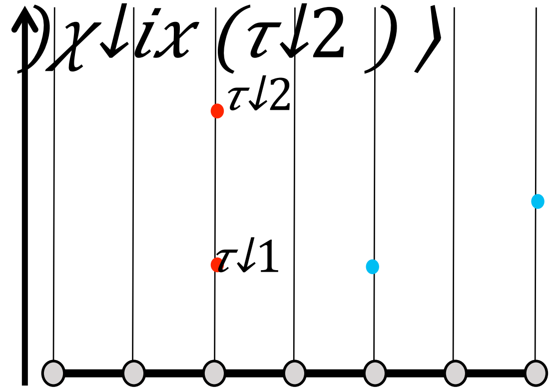
- Independent random couplings $\overline{J^2_{jklm,x}}=\frac{3!J_0^2}{N^3},\quad \overline{J'^2_{jklm,x}}=\frac{J_1^2}{N^3}$



Gu, XLQ, Stanford '16

Large-N solution

- $G_x(\tau_1,\tau_2)=\frac{1}{N}\langle\sum_i \chi_{ix}(\tau_1)\chi_{ix}(\tau_2)\rangle$ as order parameter



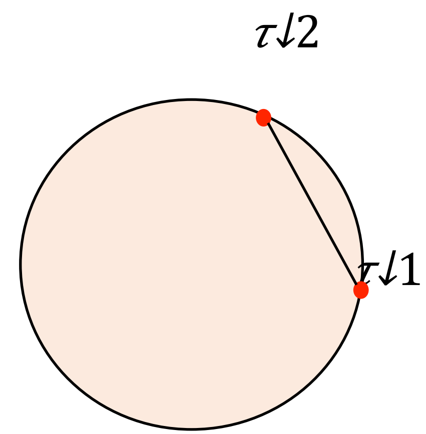
- A "dynamical mean-field" controlled by large-N
- Local criticality:

$$G_x(\tau_1,\tau_2)\propto\left|\sin\left(\frac{\pi}{\beta}(\tau_1-\tau_2)\right)\right|^{-2\Delta}.$$

- $\Delta=1/2$.
- No fermion correlation between different sites.

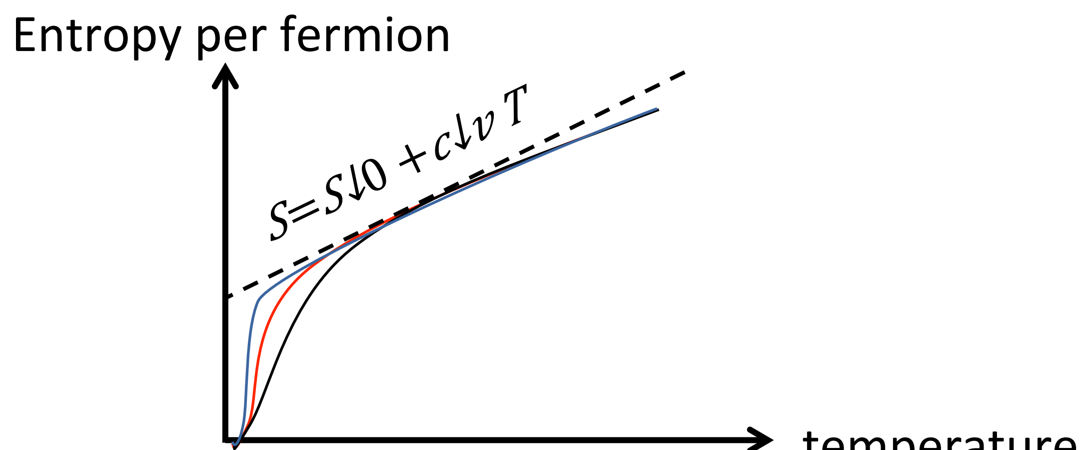
(Sachdev, Ye; Parcollet, Georges)

- At low temperature $G(\omega)\propto|\omega|^{-1/2}$

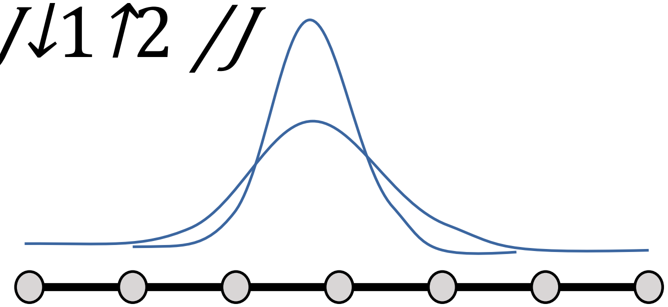


Properties of generalized SYK model

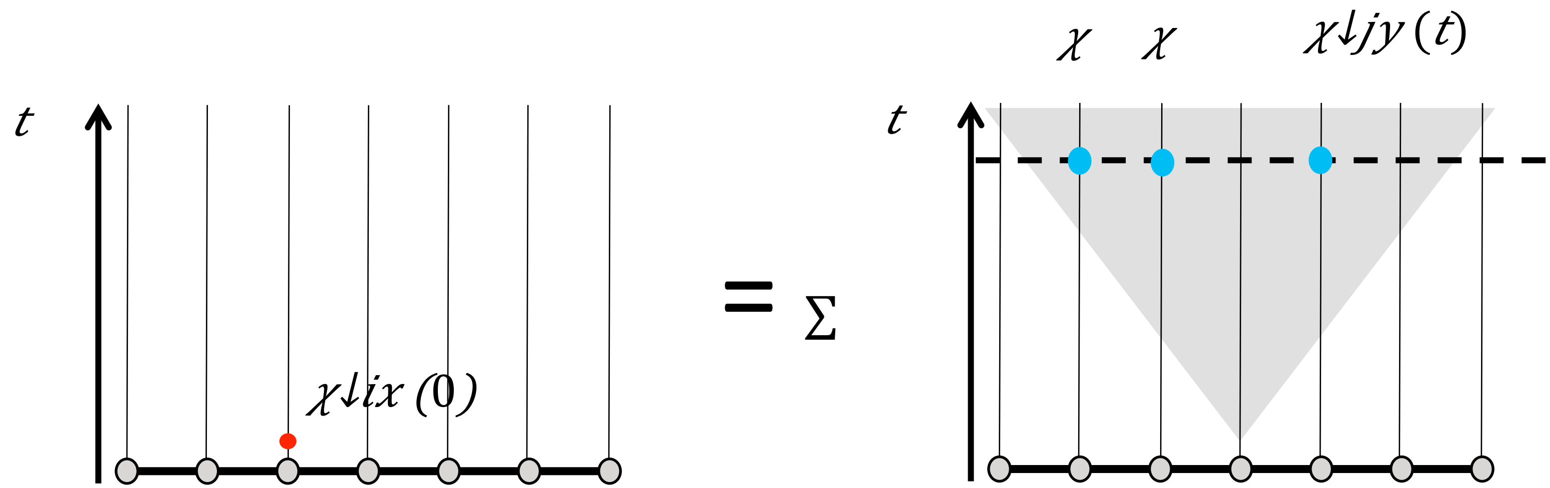
- Zero temperature entropy $S(T\to 0)=S_0$ finite in large N limit.
- A lot of low energy degrees of freedom



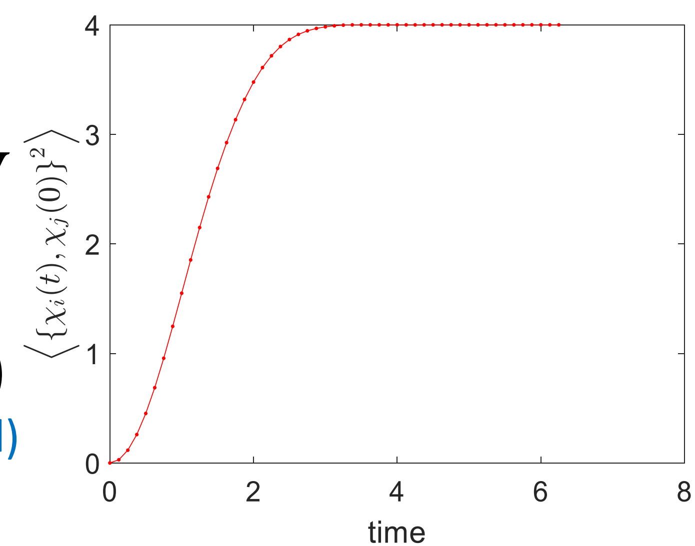
- Energy diffusion $D\propto J_1^2/J$



Chaotic dynamics

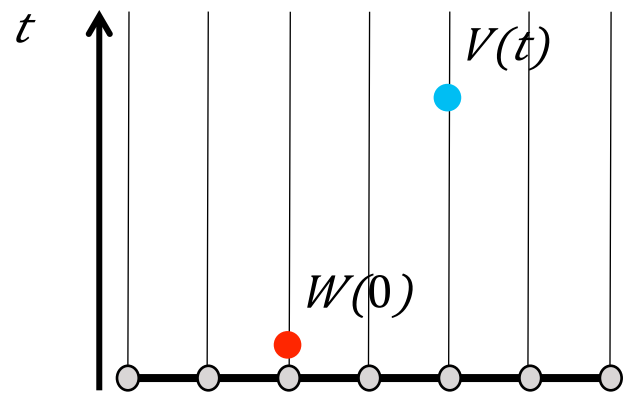


- Interacting dynamics evolves a single fermion to multi-fermion states
- Measure: size of the anti-commutator $\langle \{\chi_j^y(t), \chi_i^x(0)\}^2 \rangle_\beta \propto 1/N\, e^{\lambda(t-|x-y|/v_B)}$
- $\lambda = 2\pi T$ Lyapunov exponent (maximal)
(Maldacena-Shenker-Stanford)
- $v_B = \sqrt{D\lambda}$ butterfly velocity



Commutator growth and Lyapunov

- In more general systems, chaotic dynamics can be characterized by growth of commutator or anti-commutator:
- $\langle [V(t), W(0)]^2 \rangle_\beta$
- This is the many-body generalization of Lyapunov exponent



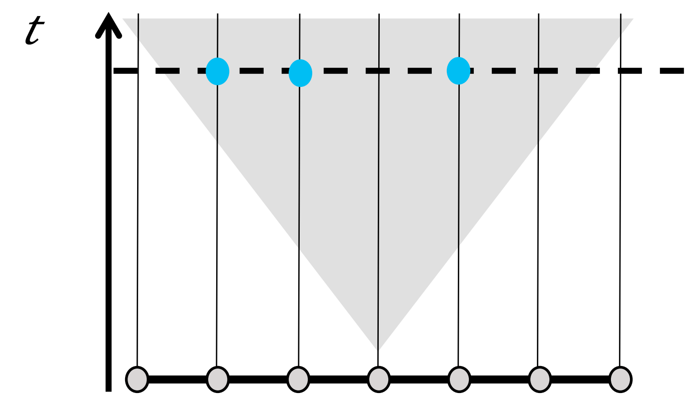
- $-i[x_i(t), p_j(0)] \to \{x_i(t), p_j(0)\}_P = \partial x_i(t)/\partial x_j(0) \propto e^{\lambda t}$

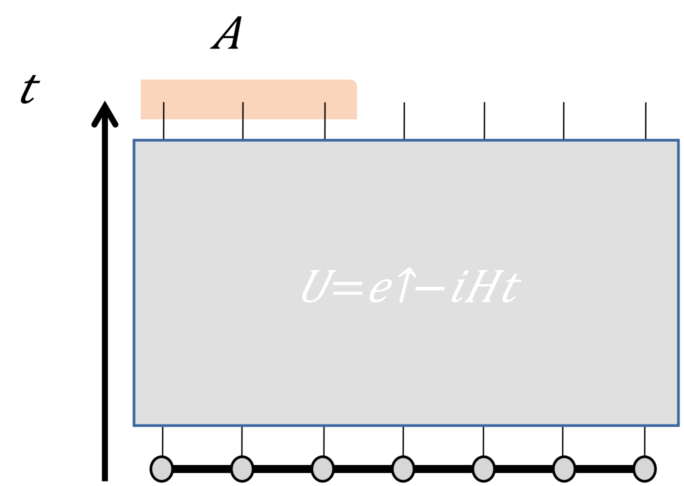
Larkin, Ovchinnikov 1969

Shenker-Stanford '13-14, Kitaev '14, Roberts-Stanford '15

Quench and thermalization

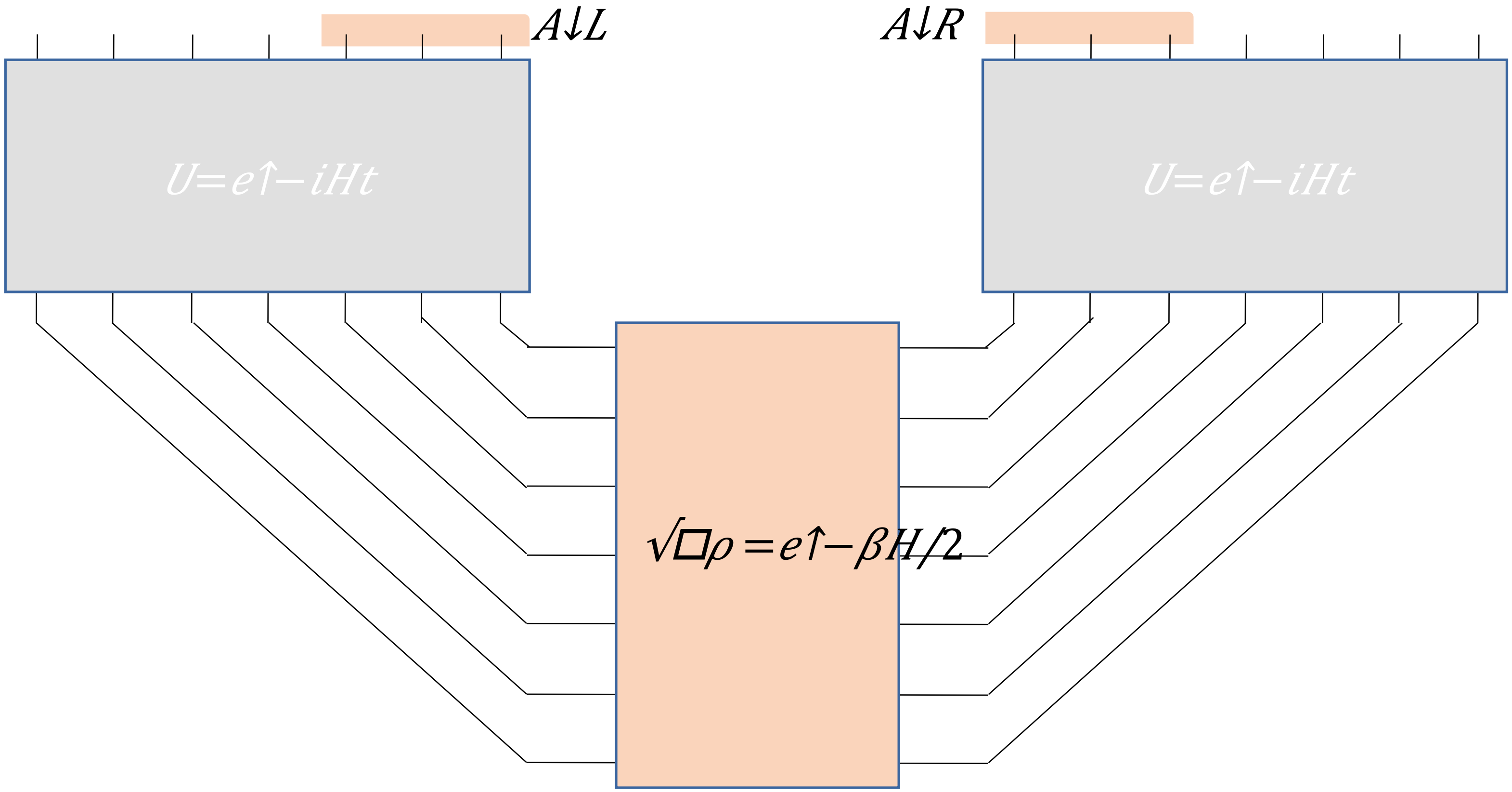
- Usually chaos implies thermalization
- Operator spreads to the whole system in time L/v_B
- Does the SYK chain thermalize in that time?
- Study the quench problem
- $|\Psi(t)\rangle = e^{-iHt}|\Psi_i\rangle$





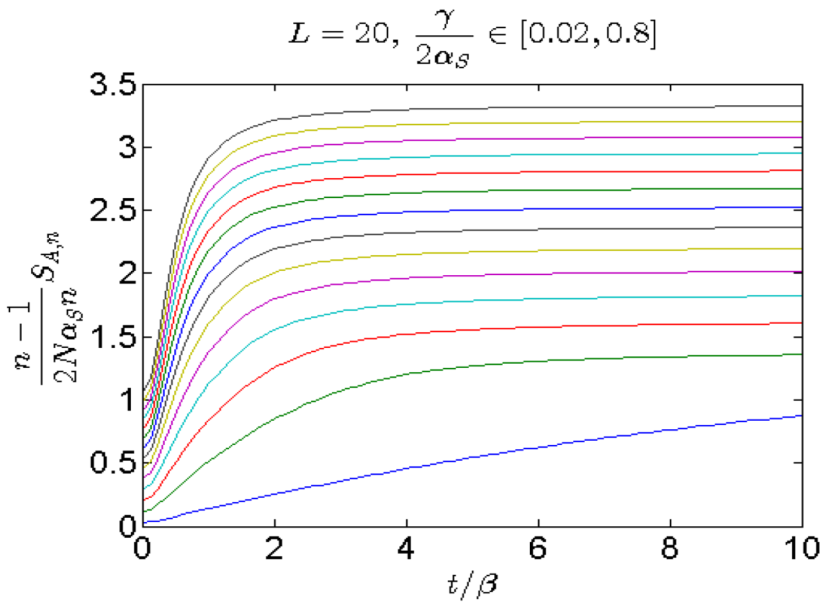
The thermal double state

- A trick to choose a simple initial state: consider two chains. $|\Psi\rangle = \sum_{nm} \left[e^{-\beta H/2}\right]_{nm} |n\rangle|m\rangle$
- Quench in the two chain (ladder) problem

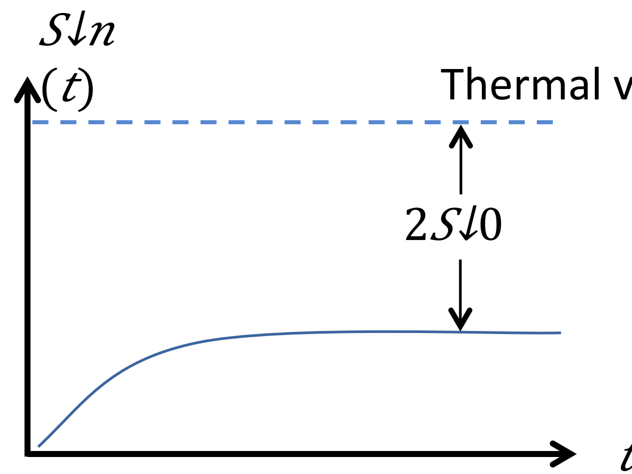


Incomplete thermalization

- Renyi entropy $S_n = \frac{1}{1-n}\log \mathrm{tr}(\rho^n)$ after quench
- Surprise: entropy does not saturate to thermal value



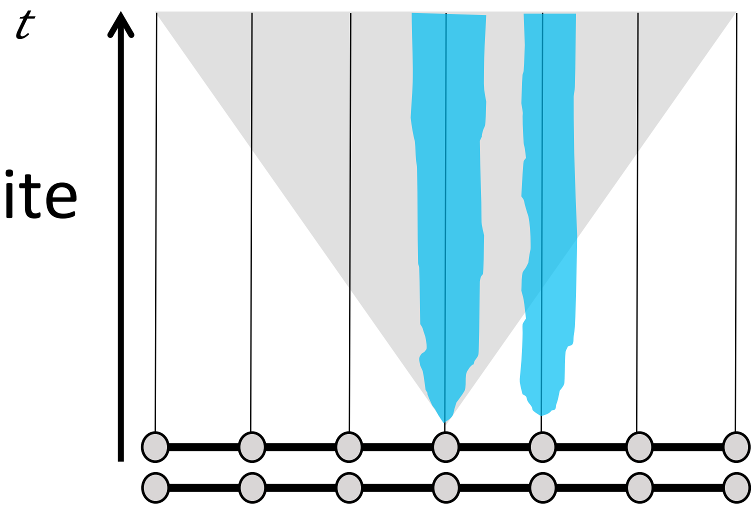
$L=\beta J,\ \gamma = J_1^2/8\pi J^2$



- Weak coupled limit $\gamma \to 0$, $S(\infty) \propto 2(S_{th} - S_0)$

Gu, Lucas, XLQ, '17

Fast and slow modes



- Non-thermalization indicates that there are localized degrees of freedom on each site
- Numer of such degrees of freedom $\sim S_0 / \log 2$
- Coexistence of fast chaotic mode that gives energy diffusion and chaos propagation and slow modes that gives zero temperature entropy
- Decoupling in the large N low temperature limit
- Finite N: thermalization in a long time?

Summary

- Generic many-body systems are chaotic
- Chaos are essential for thermalization
- Quantum entanglement provides new description to chaos and thermalization
- Solvable models can be chaotic
- Generalized SYK models consist of a coexistence of thermalizing modes and localized modes

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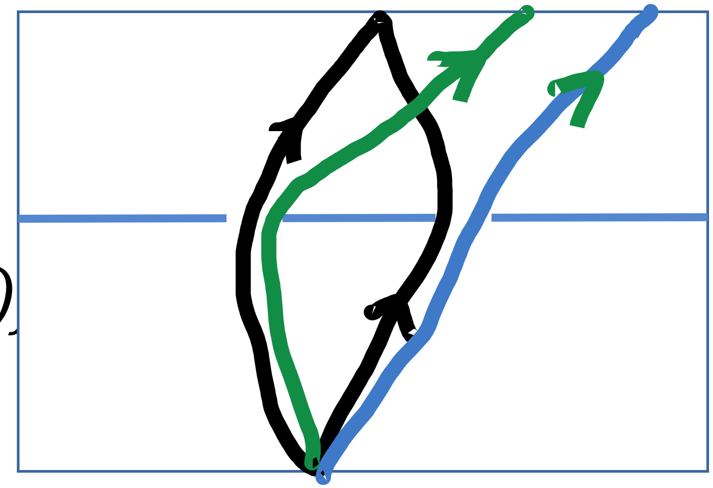
Outline

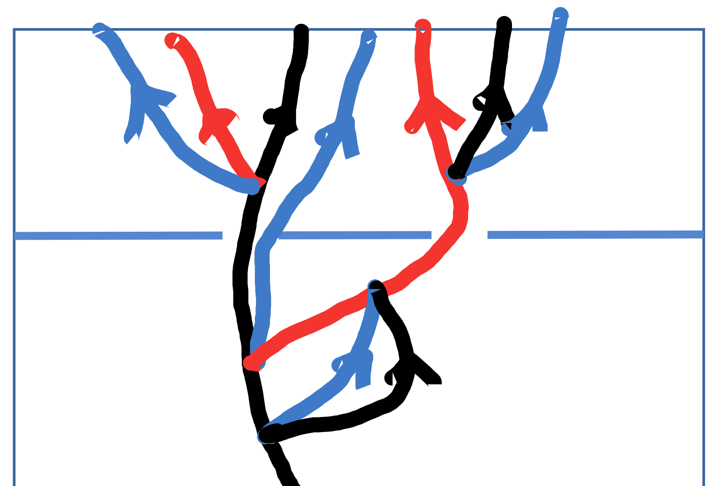
- Chaos and thermalization.
- Quantum chaos. Quantum thermalization of isolated systems.
- Entropy growth. ETH.
- Non-thermalization: MBL
- How to study this?
  - “Chaotic solvable models”. SYK model. Generalized SYK model. Energy diffusion. Coexistence of thermalization and localization. Operator growth and Lyapunov.
  - More general: Measure of chaos. Relation to thermalization. (Operator scrambling. Entanglement measure. Relation to thermalization.)

Chaos and operator scrambling

- Non-interacting system:
  A particle has N possible positions.
  $f_x(t)=\sum_{y=1}^{N} \phi_x(y)$



- Generic interacting system:
  A particle can decay into multi-particle states.
  Exponentially many final states in the Hilbert space.



- $f_x(t)=\phi_x(y) f_y(0)+$
  $\phi_x(y_1\, y_2\, y_3) f_{y_1}\, f_{y_2}\, f_{y_3} +\ldots$