Symmetric Surfaces of Topological Superconductor

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1819

Outline

- Introduction
  - Brief description of time reversal symmetric topological superconductor.
- Coupled wire model
  - Why coupled wire model ?
  - Coupled wire model for the gapless surface of topological superconductor.
- Gapping interaction
  - General gapping term for any N.
  - Show explicitly for odd N case.
- Topological order of gapped surface
  - Use bulk-boundary correspondence to get anyon content of the surface.
  - 32-fold topological order

BCS Theory

- Hamiltonian, $$\mathcal{H} = \underbrace{\sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}}_{\text{KE}} + \underbrace{\frac{1}{N}\sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}}_{\text{Cooper pairing}}$$

- Mean field approximation, $$AB \approx \langle A\rangle B + \langle B\rangle A + const$$

- Define $$\Delta_{\mathbf{k}} = -\frac{1}{N}\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}\rangle$$

- Mean field BCS Hamiltonian $$\mathcal{H} = \frac{1}{2}\sum_{\mathbf{k}} \begin{pmatrix} c^\dagger_{\mathbf{k}} & c_{-\mathbf{k}} \end{pmatrix} \underbrace{\begin{pmatrix} \varepsilon(\mathbf{k}) & \Delta(\mathbf{k}) \\ \Delta^*(\mathbf{k}) & -\varepsilon(\mathbf{k}) \end{pmatrix}}_{H_{BdG}} \begin{pmatrix} c_{\mathbf{k}} \\ c^\dagger_{-\mathbf{k}} \end{pmatrix}$$

Nambu basis, $\xi^\dagger_{\mathbf{k}} = \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} & c^\dagger_{\mathbf{k}\downarrow} & c_{-\mathbf{k}\downarrow} & -c_{-\mathbf{k}\uparrow} \end{pmatrix}$

BCS Theory: Symmetry

Anti-unitary symmetries of BdG Hamiltonian

- Particle-Hole Symmetry (PHS) , C $\quad c_{\mathbf{k}\uparrow} \rightarrow c^{\dagger}_{\mathbf{k}\uparrow}$

$$C = \sigma_y \tau_y K \qquad\qquad \{C, H_{BdG}(\mathbf{k})\} = 0$$

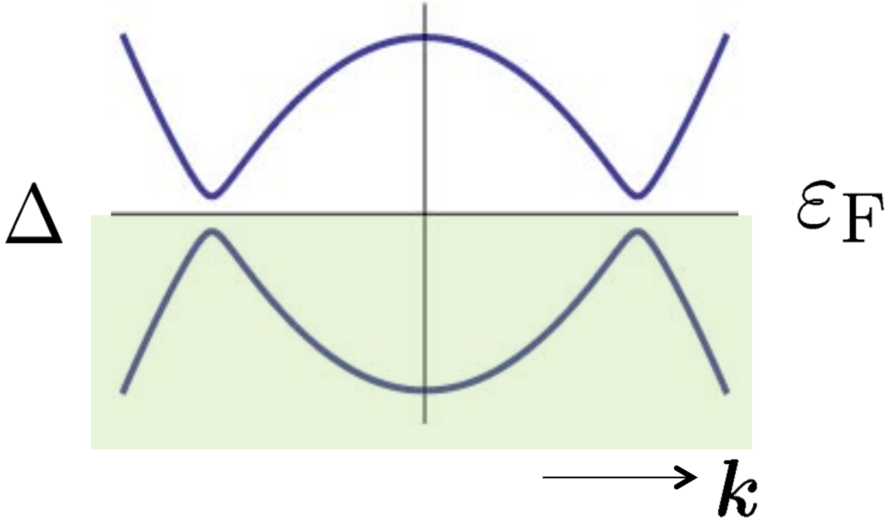
- In some cases, Time Reversal Symmetry (TRS) , T $\quad c_{\mathbf{k}\uparrow} \rightarrow c_{-\mathbf{k}\downarrow}$

$$T = i\sigma_y K \qquad\qquad [T, H_{BdG}(\mathbf{k})] = 0$$

- We can define Chiral Symmetry (CS), $\Pi = iCT \quad \{\Pi, H_{BdG}(\mathbf{k})\} = 0$

BCS Theory (contd.)

- BdG Transformation $\gamma_{\mathbf{k}\uparrow} = u^*_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow}$
- Diagonalizing $\mathcal{H}_{BCS} = \sum_{\mathbf{k},\sigma} \sqrt{\epsilon^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2}\ \gamma^\dagger_{\mathbf{k}\sigma}\gamma_{\mathbf{k}\sigma}$



- Topological superconductor has non-trivial band-structure.
- Depending on whether symmetries squares to $\pm$ identity we have different class of topological superconductor.

Classification of Topological Insulators and Superconductors

		TRS	PHS	CS	d=1	d=2	d=3
Standard (Wigner-Dyson)	A (unitary)	0	0	0	-	$\mathbb{Z}$	-
	AI (orthogonal)	+1	0	0	-	-	-
	AII (symplectic)	-1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Chiral (sublattice)	AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
	BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-
	CII (chiral symplectic)	-1	-1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
BdG	D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
	C	0	-1	0	-	$\mathbb{Z}_2$	-
	$\boxed{\text{DIII}}$	-1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\boxed{\mathbb{Z}}$
	CI	+1	-1	1	-	-	$\mathbb{Z}$

[Schnyder, et.al., 08] [Kitaev, 08] [Qi et.al., 09]

Topological Superconductor

Class DIII topological superconductor:

$$H_{BdG} = X(\mathbf{k}) \otimes \tau_x + Z(\mathbf{k}) \otimes \tau_z$$

- 3D Brillouin zone $\xrightarrow{q(\mathbf{k})}$ $GL(n, \mathbb{C}) \simeq U(n)$

$$q(\mathbf{k}) = X(\mathbf{k}) + iZ(\mathbf{k}), \text{ s.t. } det(q) \neq 0$$

- Winding number,

$$\nu[q] = \int \frac{d^3k}{24\pi^2} \epsilon^{\mu\nu\rho} Tr\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q)\right] \in \mathbb{Z}$$

Because $\pi_3(U(n)) = \mathbb{Z}$

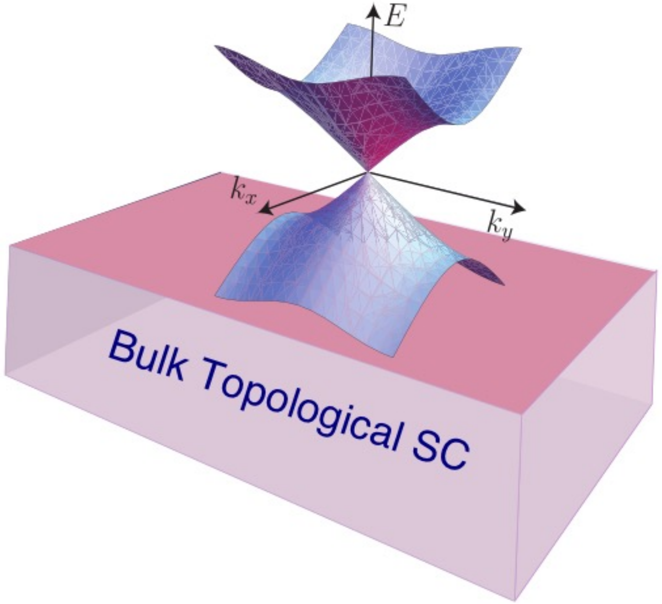
Topological Superconductor(contd.)

- Topological invariant is the winding number, N that takes integer value.

- How does the winding number, N appear in transport phenomena?

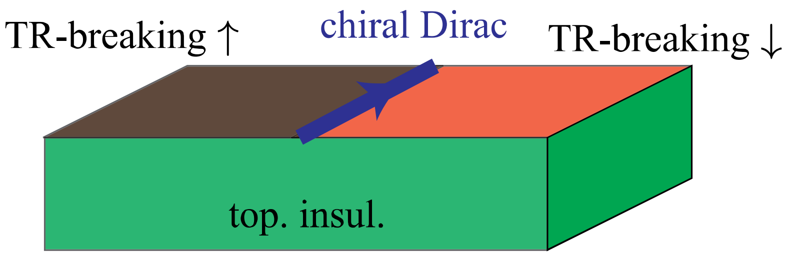
  N # of gapless Majorana cones on the surface.

- Adding any TR symmetric single-body perturbation will not open a gap as long as bulk gap is not closed.



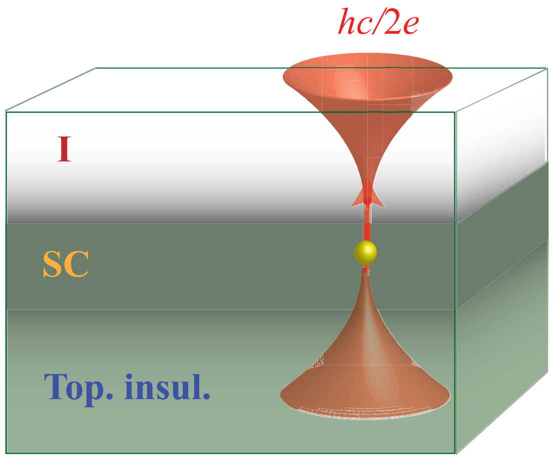
Gapped surfaces of Top. Insulator and Superconductor

- TR-breaking surface of TI

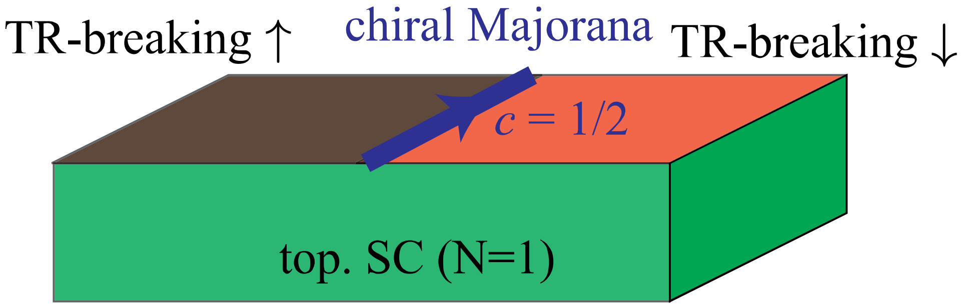


TR breaking through magnetic field

- Charge breaking surface of TI



- TR-breaking surface of TSc



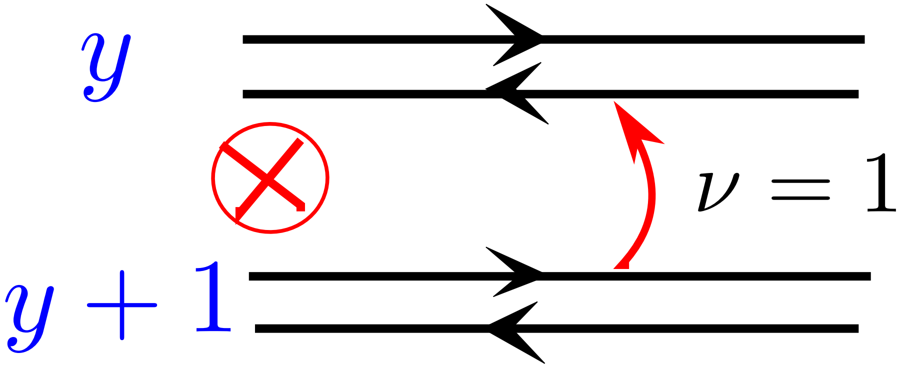
TR breaking when pairing phase $\neq 0$ or π

What about gapped surface that are symmetric ?

Why coupled wire model ?

- This method allows us to write simple Hamiltonian in terms of local degrees of freedom rather than field theory description.
- Hamiltonian can be transformed using bosonization and can be exactly solved.

2D integer quantum Hall state

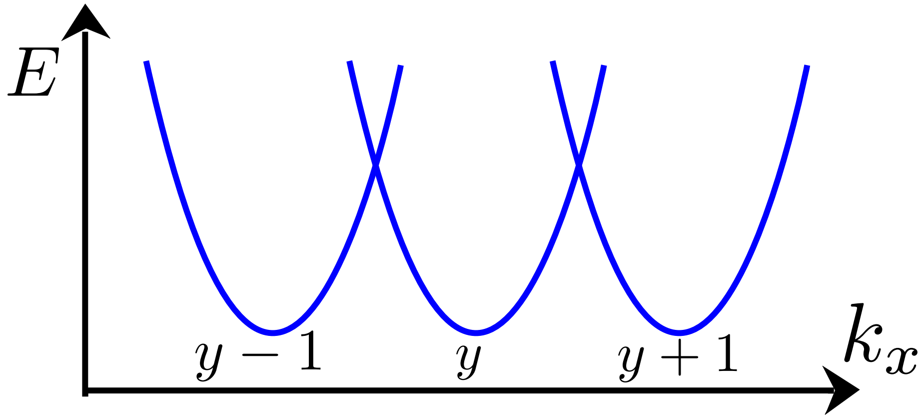


$$\mathcal{O} = \int dx\ c_y^{\dagger L} c_{y+1}^R$$

[Sondhi, Yang 2001] ; [Vishwanath, Carpentier 2001]

In Landau guage $\mathbf{A} = -By\hat{x}$

$$E_y(k_x) = \frac{\hbar^2}{2m}\left(k_x - \frac{eBy}{\hbar c}\right)^2$$



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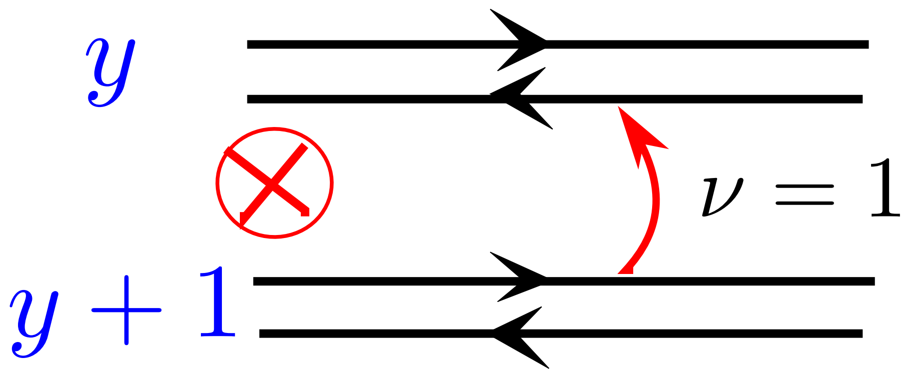

$$\mathcal{O} = \int dx\; c^{\dagger L}_{y} c^{R}_{y+1}$$

[Sondhi, Yang 2001] ; [Vishwanath, Carpentier 2001]

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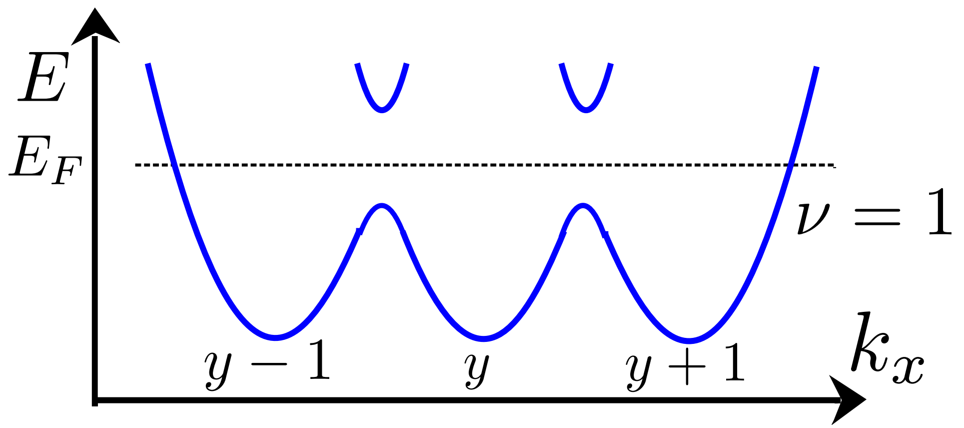
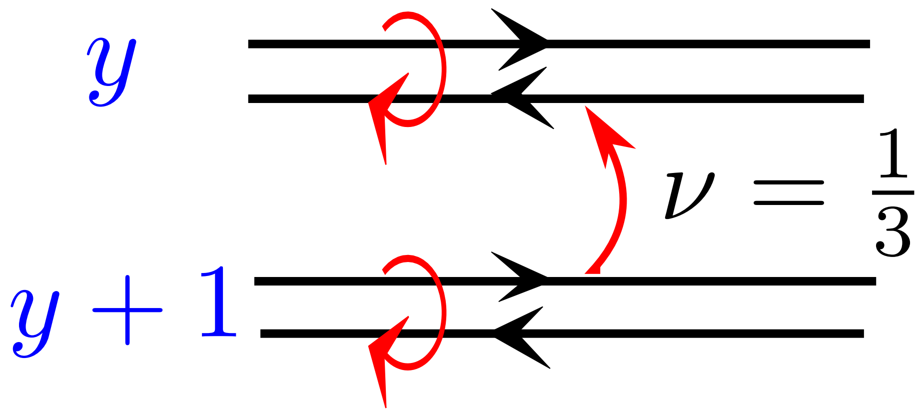
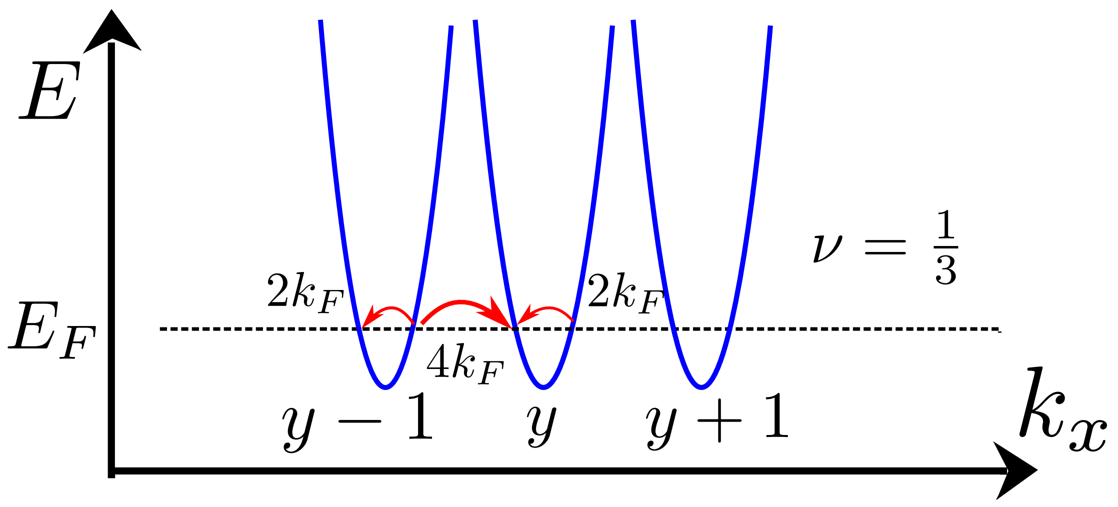


Why coupled wire model ?

- How about fractional quantum hall effect? [Kane, Mukhopadyay, Lubensky 2002]

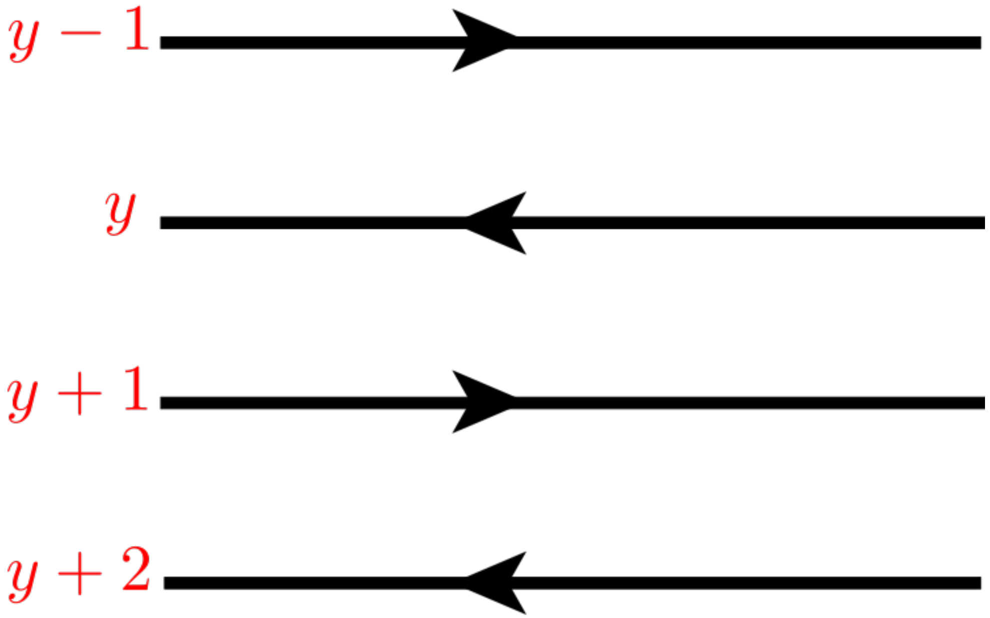




- We can also construct coupled wire model for the 2D surface of 3D topological states.
- Such model was proposed for 3D topological insulator

D. F. Mross, A. Essin, and J. Alicea, Phys. Rev. X **5**, 011011 (2015)

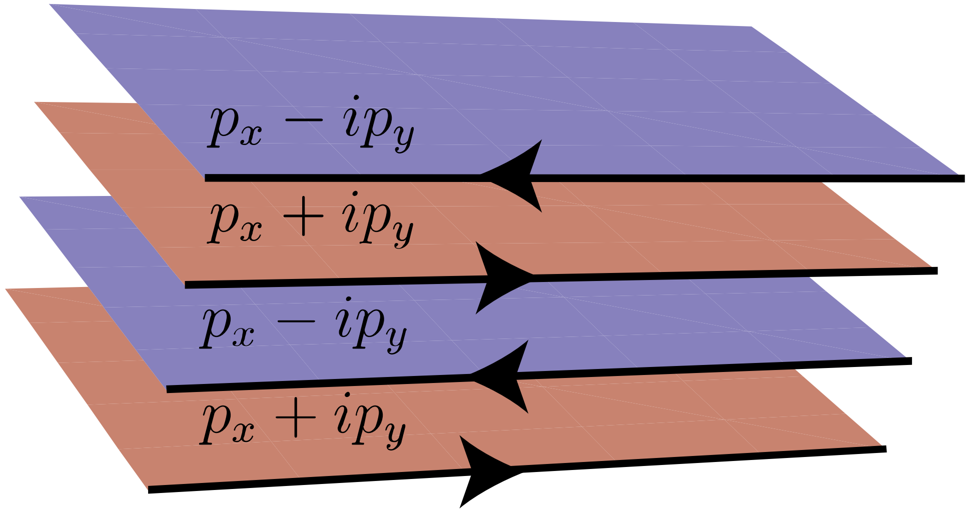
Coupled Wire Model

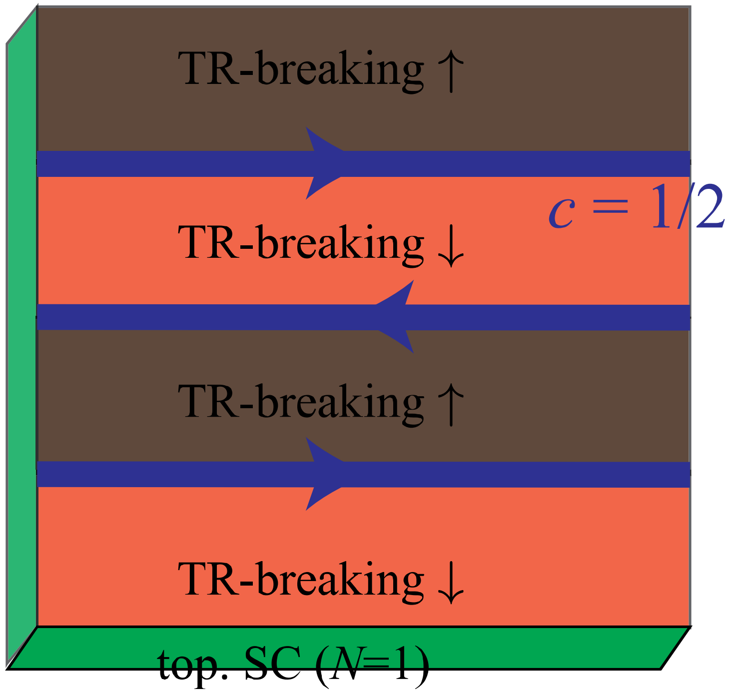


Majorana fermions

$$\boldsymbol{\psi}_y = (\psi_y^1, \psi_y^2, \cdots \psi_y^N)$$

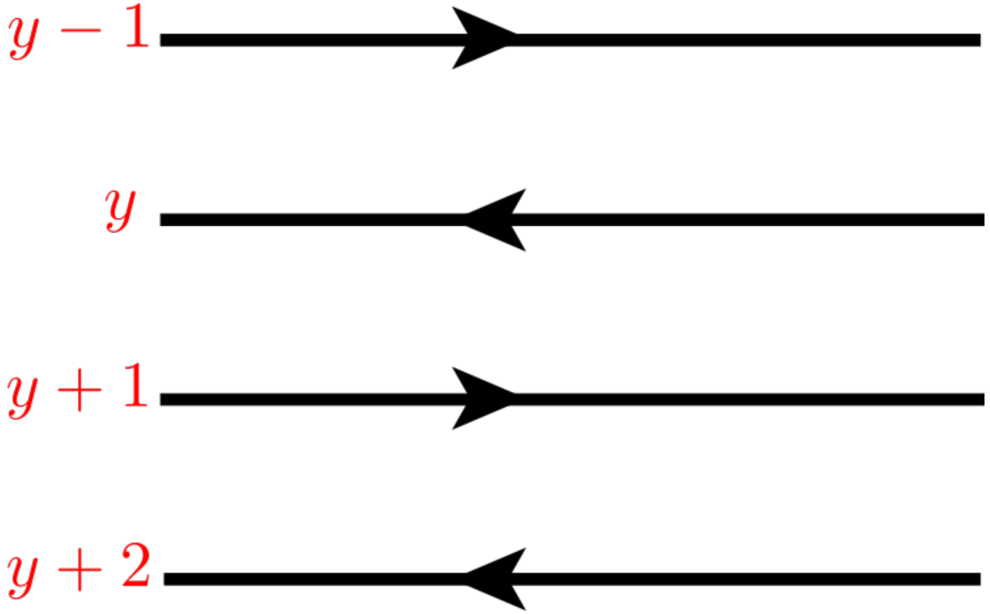
Coupled Wire Model (contd.)





Stack of alternating p_x+ip_y

Coupled Wire Model (contd.)

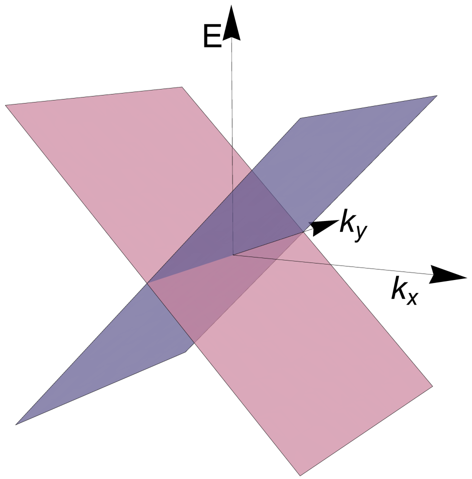


Majorana fermions

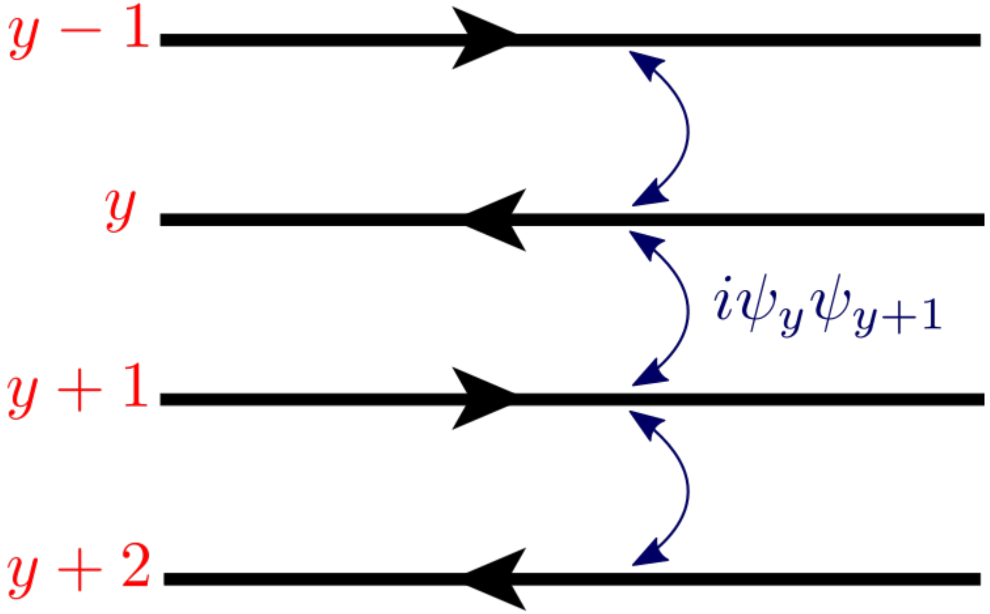
$$\boldsymbol{\psi}_y = \left(\psi_y^1, \psi_y^2, \cdots \psi_y^N\right)$$

Kinetic energy for motion along the wires

$$\mathcal{H}_0 = \sum_{y=-\infty}^{\infty} i v_{\mathsf{x}} (-1)^y \boldsymbol{\psi}_y^T \partial_x \boldsymbol{\psi}_y$$



Coupled Wire Model (contd.)



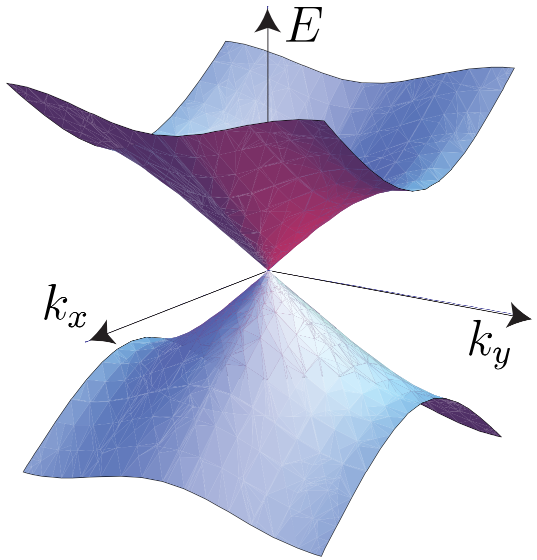
$$\mathcal{H}_0 = \sum_{\mathbf{k}} \boldsymbol{\xi}_{\mathbf{k}}^\dagger H_{\mathrm{BdG}}^0(\mathbf{k})\boldsymbol{\xi}_{\mathbf{k}}, \qquad \boldsymbol{\xi}_{\mathbf{k}} = (c_{\mathbf{k}}^a, {c_{-\mathbf{k}}^a}^\dagger)^T$$

Nambu basis

Majorana fermions

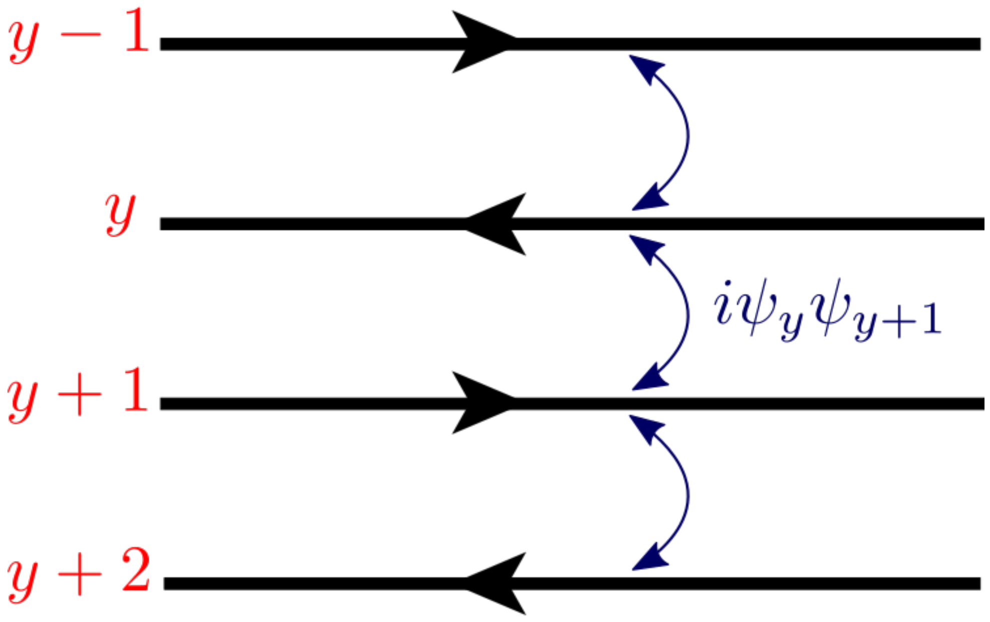
$$\boldsymbol{\psi}_y = \left(\psi_y^1, \psi_y^2, \cdots \psi_y^N\right)$$

$$\mathcal{H}_0 = \sum_{y=-\infty}^{\infty} i v_{\mathrm{x}}(-1)^y \boldsymbol{\psi}_y^T \partial_x \boldsymbol{\psi}_y + i v_{\mathrm{y}} \boldsymbol{\psi}_y^T \boldsymbol{\psi}_{y+1}$$



$$H_{\mathrm{BdG}}^0(\mathbf{k}) = 2v_{\mathrm{x}} k_x \tau_x + v_{\mathrm{y}}\left[-\sin k_y \tau_y + (1-\cos k_y)\tau_z\right]$$

Coupled Wire Model (contd.)



Majorana fermions

$$\boldsymbol{\psi}_y = \left(\psi_y^1, \psi_y^2, \cdots \psi_y^N\right)$$

$$\mathcal{H}_0 = \sum_{y=-\infty}^{\infty} i v_{\mathsf{x}}(-1)^y \boldsymbol{\psi}_y^T \partial_x \boldsymbol{\psi}_y + i v_{\mathsf{y}} \boldsymbol{\psi}_y^T \boldsymbol{\psi}_{y+1}$$

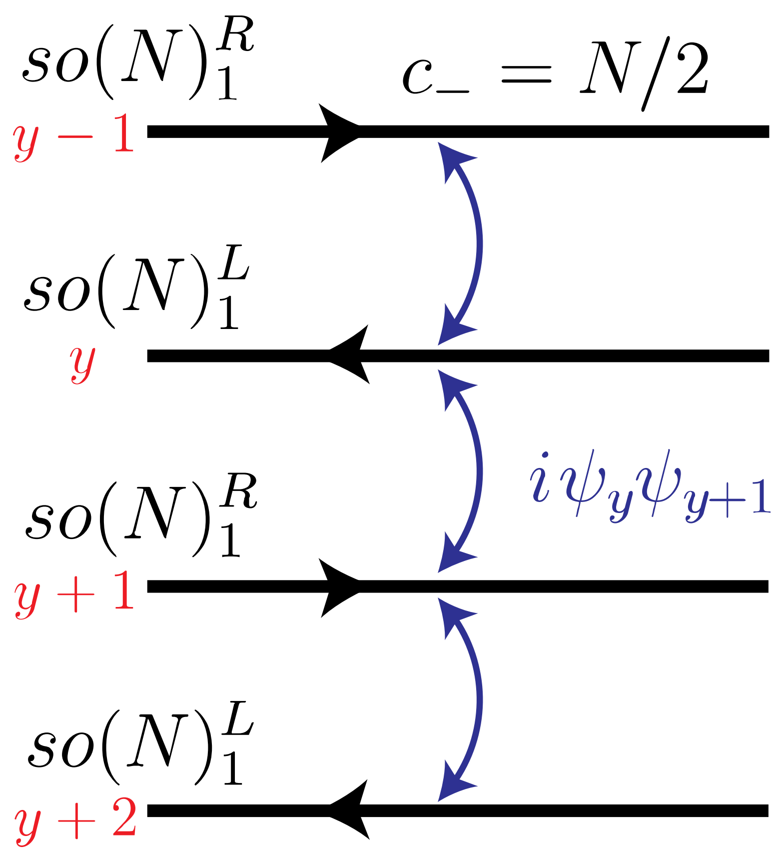
- Antiferromagnetic TRS,

$$T : \psi_y \to (-1)^y \psi_{y+1}$$

T^2 translates by two lattice unit in y direction (non-symomorphic symmetry)

New classification $\mathbb{Z}_2$ instead of $\mathbb{Z}$

SO(N)$_1$ WZW CFT



- SO(N) symmetry

$$\psi_y^a \to O_a^b \psi_y^b, \quad \boldsymbol{\psi}_y = \left(\psi_y^1, \psi_y^2, \cdots \psi_y^N\right)$$

- Conserved current

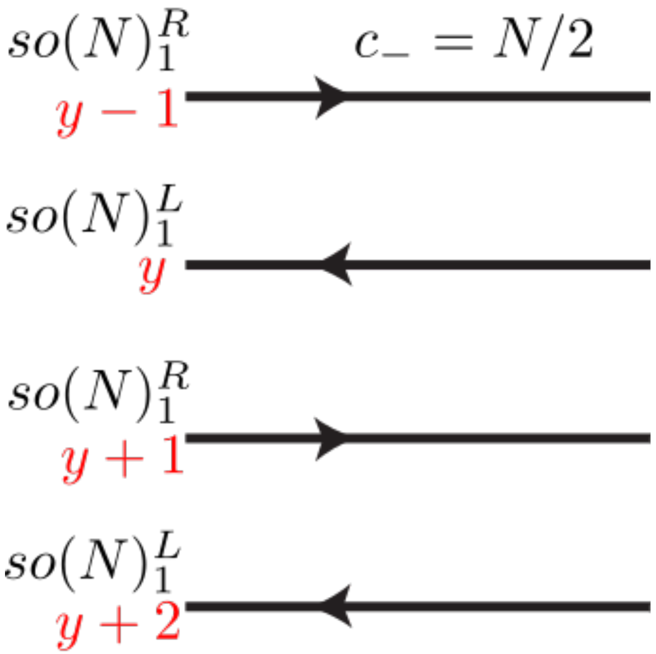
$$J^\beta(z) = \frac{i}{2} \sum_{a,b} \psi_a(z) t_{ab}^\beta \psi_b(z)$$

- Energy momentum tensor operator product expansion (OPE) gives the central charge c_- = N/2.

$$c_- = N/2$$

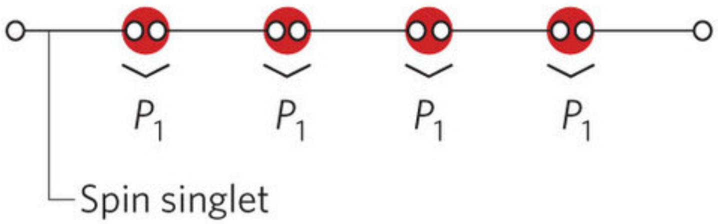
$$I_T \approx c_- \frac{\pi^2 k_B^2}{6h} T^2$$

Gapping interaction



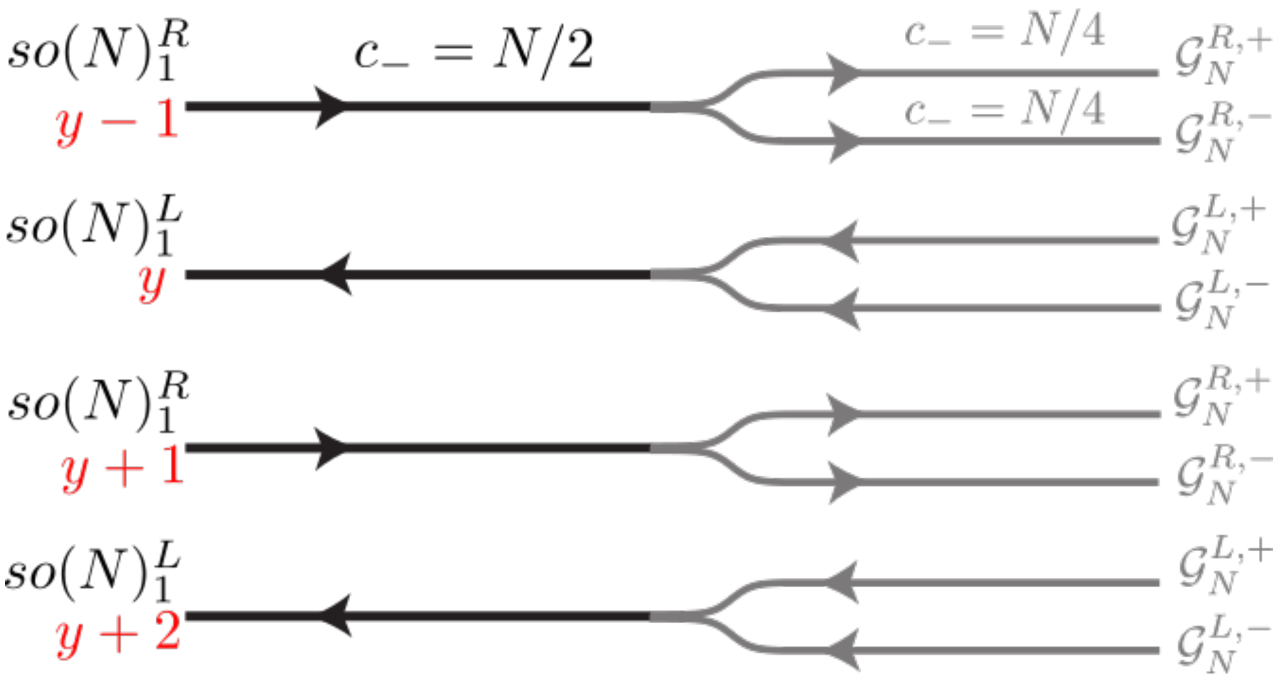
Fractionalization $\boxed{so(N)_1 \supseteq \mathcal{G}_N^+ \times \mathcal{G}_N^-}$

AKLT spin chain



Picture from doi:10.1038/nphys1829

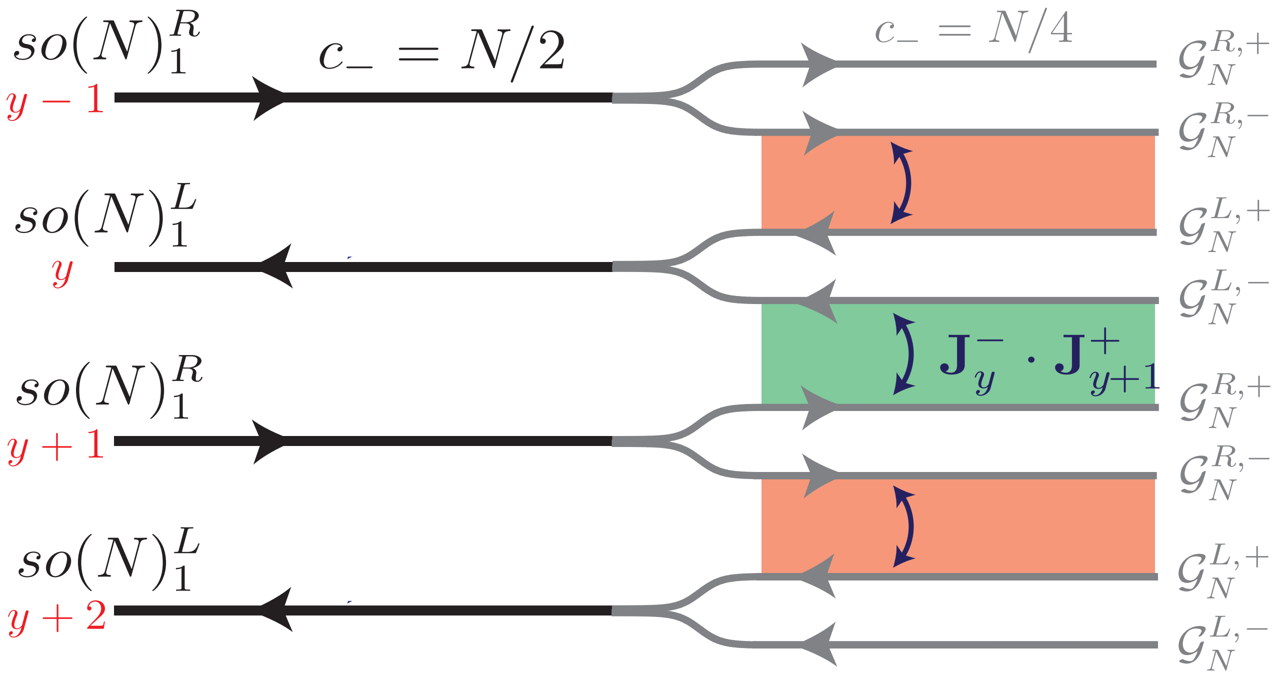
Gapping interaction



Fractionalization $\boxed{so(N)_1 \supseteq \mathcal{G}_N^+ \times \mathcal{G}_N^-}$

$$\mathcal{G}_N = \begin{cases} so(r)_1, & \text{for EVEN } N = 2r \\ so(3)_3 \otimes so(r)_1, & \text{for ODD } N = 9 + 2r \end{cases}$$

Gapping interaction (Contd.)



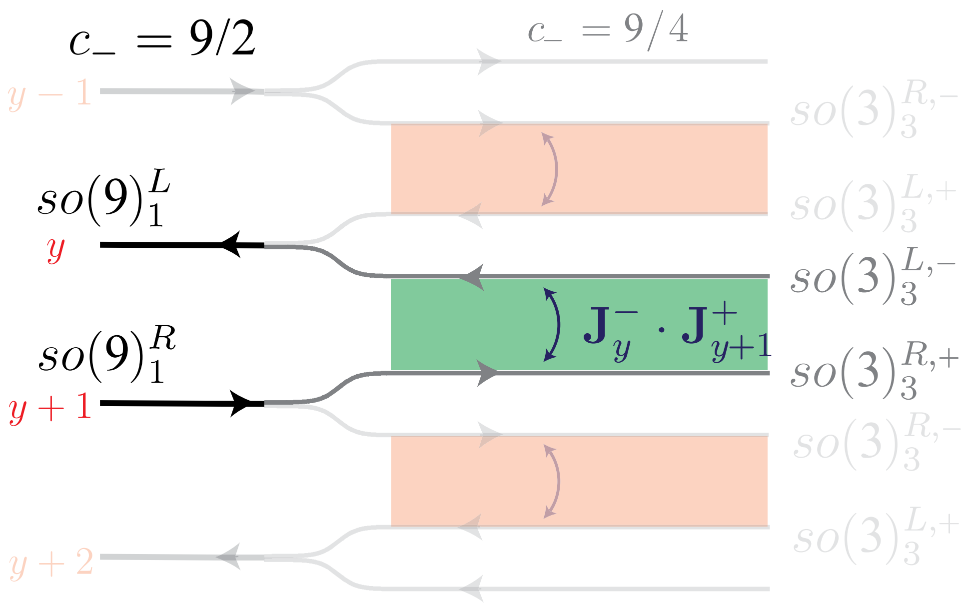
Sahoo, Zhang, Teo
arXiv:1509.07133

Fractionalization $\boxed{so(N)_1 \supseteq \mathcal{G}_N^+ \times \mathcal{G}_N^-}$

4-Fermion gapping interaction

$$\boxed{\mathcal{H}_{\text{int}} = u \sum_{y=-\infty}^{\infty} \mathbf{J}^y_{\mathcal{G}_N^-} \cdot \mathbf{J}^{y+1}_{\mathcal{G}_N^+}}$$

Gapping Odd N Majorana's



Special case: $N = 9 = 3^2$

Conformal embedding (level rank duality):

$$\boxed{so(9)_1 \supseteq so(3)_3 \times so(3)_3}$$

Generator of so(3): $\quad \boldsymbol{\Sigma} = (\Sigma_x, \Sigma_y, \Sigma_z)$

$$\Sigma_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \Sigma_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Sigma_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Embedding

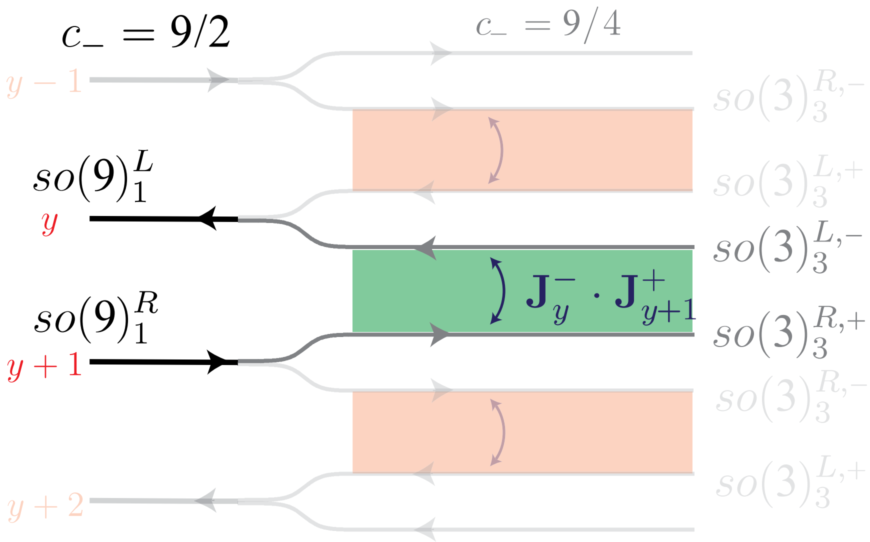
$$\boldsymbol{\Sigma}^+ = \boldsymbol{\Sigma} \otimes \mathbb{1}_3, \quad \boldsymbol{\Sigma}^- = \mathbb{1}_3 \otimes \boldsymbol{\Sigma}.$$

$$\mathbf{J}_{so(3)^\pm_3}(z) = \frac{i}{2}\psi^a(z)\boldsymbol{\Sigma}^\pm_{ab}\psi^b(z)$$

$$J^+_x = i(\psi^{23} + \psi^{56} + \psi^{89}), \quad J^-_x = i(\psi^{47} + \psi^{58} + \psi^{69})$$
$$J^+_y = i(\psi^{13} + \psi^{46} + \psi^{79}), \quad J^-_y = i(\psi^{17} + \psi^{28} + \psi^{39})$$
$$J^+_z = i(\psi^{12} + \psi^{45} + \psi^{78}), \quad J^-_z = i(\psi^{14} + \psi^{25} + \psi^{36})$$

for $\psi^{ab} = \psi^a\psi^b$

Gapping Odd N Majorana's



4-fermion TR-symmetric interaction between a pair of wires:

$$\mathcal{H}_{int} = u\mathbf{J}^L_{so(3)_3^-} . \mathbf{J}^R_{so(3)_3^+}$$

Symmetric under antiferromagnetic time reversal,

$$\mathbf{J}_y^{L,+} \to \mathbf{J}_{y+1}^{R,+}$$

Marginally relevant $\quad \dfrac{du}{dl} = +4\pi\alpha u^2$

For $so(3)_3^{L,-}$

$$J_\pm^{L,-} = i\left[(\psi^{4,L} \pm i\psi^{1,L})\psi^{7,L} + (\psi^{5,L} \pm i\psi^{2,L})\psi^{8,L} + (\psi^{6,L} \pm i\psi^{3,L})\psi^{9,L}\right]$$

$$J_z^{L,-} = i\left[\psi^{1,L}\psi^{4,L} + \psi^{2,L}\psi^{5,L} + \psi^{3,L}\psi^{6,L}\right]$$

For $so(3)_3^{R,+}$

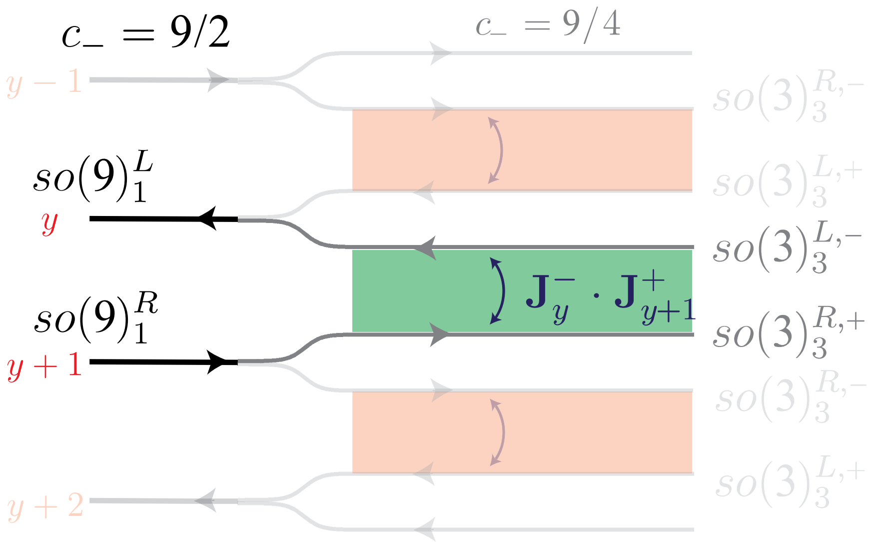
$$J_\pm^{R,+} = i\left[(\psi^{2,R} \pm i\psi^{1,R})\psi^{3,R} + (\psi^{5,R} \pm i\psi^{4,R})\psi^{6,R} + (\psi^{8,R} \pm i\psi^{7,R})\psi^{9,R}\right]$$

$$J_z^{R,+} = i\left[\psi^{1,R}\psi^{2,R} + \psi^{4,R}\psi^{5,R} + \psi^{7,R}\psi^{8,R}\right]$$

$$\mathcal{H}_{int} = J_z^{L,-}J_z^{R,+} + \frac{1}{2}\left(J_+^{L,-}J_-^{R,+} + J_-^{L,-}J_+^{R,+}\right)$$

Four fermion interaction

Gapping Odd N Majorana's



Interaction Hamiltonian is a sum of pairwise interactions.

These terms commute i.e $\left[\mathbf{J}_y^- . \mathbf{J}_{y+1}^+ , \mathbf{J}_{y+1}^- . \mathbf{J}_{y-1}^+\right] = 0$

This Hamiltonian is exactly solvable.

$$\sqrt{2}c^{1,L} \sim e^{i\phi^{1,L}} \qquad \sqrt{2}c^{2,L} \sim e^{i\phi^{2,L}} \qquad \sqrt{2}c^{3,L} \sim e^{i\phi^{3,L}}$$

For $so(3)_3^{L,-}$

$$J_\pm^{L,-} = i\left[(\psi^{4,L} \pm i\psi^{1,L})\psi^{7,L} + (\psi^{5,L} \pm i\psi^{2,L})\psi^{8,L} + (\psi^{6,L} \pm i\psi^{3,L})\psi^{9,L}\right]$$

$$J_z^{L,-} = i\left[\psi^{1,L}\psi^{4,L} + \psi^{2,L}\psi^{5,L} + \psi^{3,L}\psi^{6,L}\right]$$

For $so(3)_3^{R,+}$

$$J_\pm^{R,+} = i\left[(\psi^{2,R} \pm i\psi^{1,R})\psi^{3,R} + (\psi^{5,R} \pm i\psi^{4,R})\psi^{6,R} + (\psi^{8,R} \pm i\psi^{7,R})\psi^{9,R}\right]$$

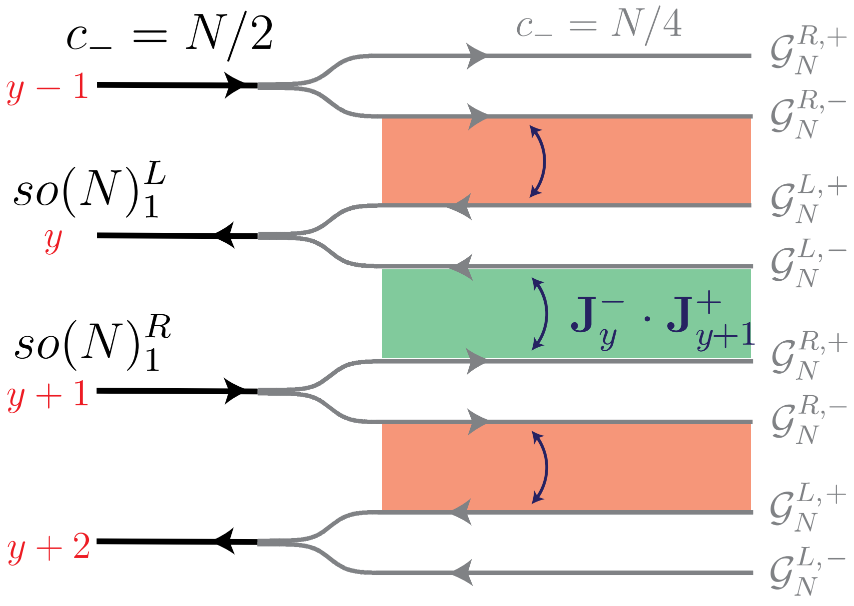
$$J_z^{R,+} = i\left[\psi^{1,R}\psi^{2,R} + \psi^{4,R}\psi^{5,R} + \psi^{7,R}\psi^{8,R}\right]$$

$$\mathcal{H}_{int} = J_z^{L,-}J_z^{R,+} + \frac{1}{2}\left(J_+^{L,-}J_-^{R,+} + J_-^{L,-}J_+^{R,+}\right)$$

Four fermion interaction

Gapping Odd N Majorana's

$\boxed{N = 9 + 2r}$



$$so(N)_1$$
$$\downarrow$$
$$so(9)_1 \times so(2r)_1$$
$$\swarrow \qquad \searrow$$
$$\underset{\mathcal{G}_N^+}{so(3)_3 \times so(r)_1} \qquad \underset{\mathcal{G}_N^-}{so(3)_3 \times so(r)_1}$$

Negative r = reverse propagating channels

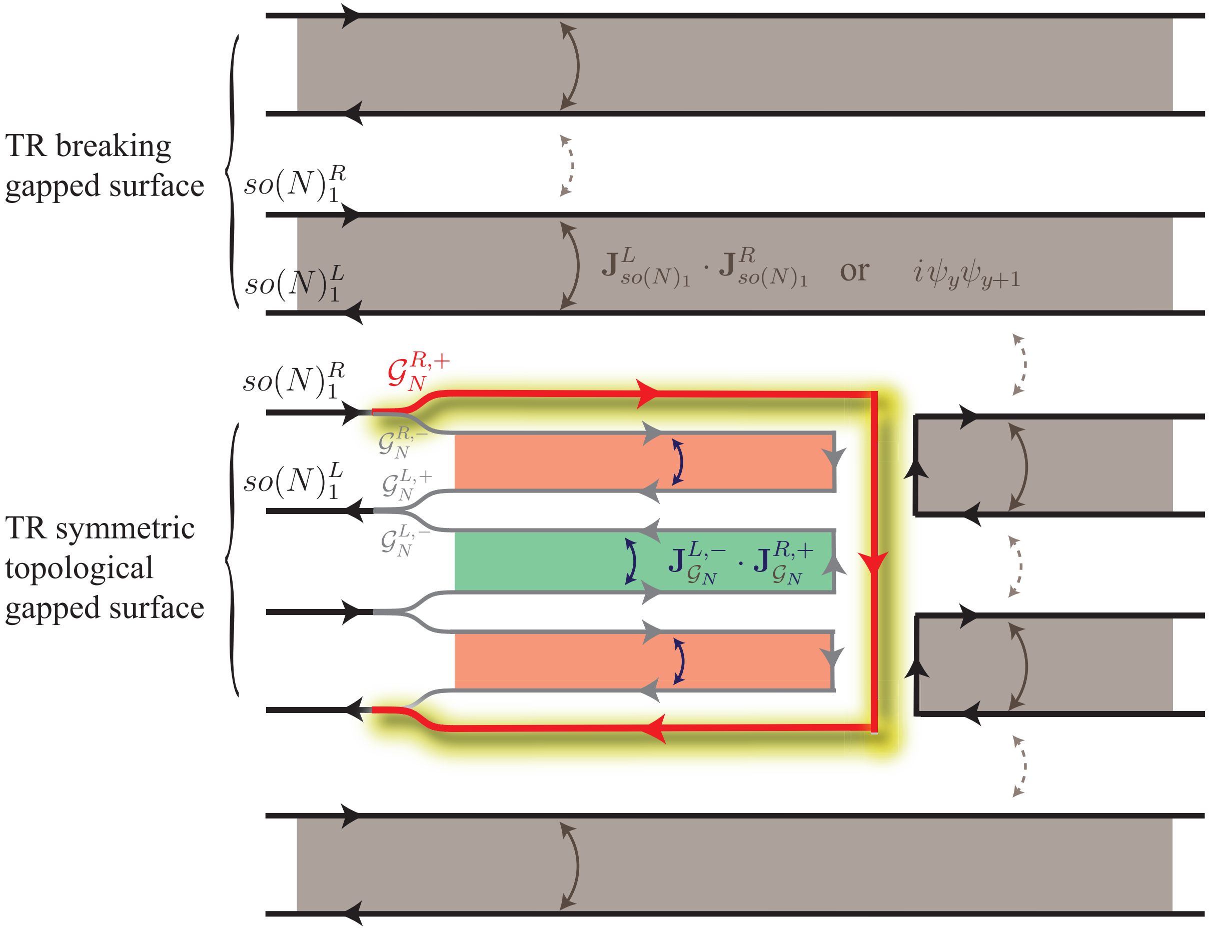
$$\mathcal{H}_{\text{int}} = u\mathbf{J}_{\mathcal{G}_N^+} \cdot \mathbf{J}_{\mathcal{G}_N^-}$$
$$= u\mathbf{J}_{so(3)_3^+} \cdot \mathbf{J}_{so(3)_3^-} + u\mathbf{J}_{so(r)_1^+} \cdot \mathbf{J}_{so(r)_1^-}$$

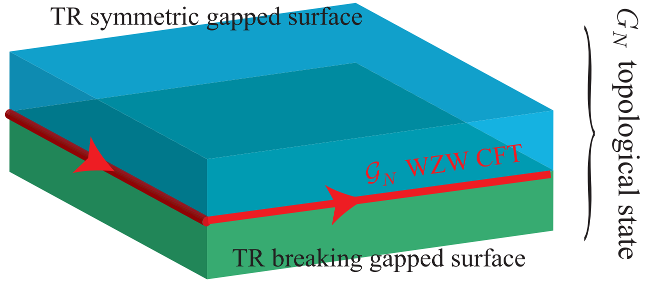
Bulk-Boundary correspondence

Bulk topological theory	Boundary conformal field theory
• Quasi-particles	• Primary fields
• Fusion	• Operator product expansion
• Exchange statistics	• Scaling dimension
• Braiding	• Modular transformations

[Read, Moore 93']

Topological order in slab geometry





- Bulk is topological superconductor
- Upper surface is TR symmetric gapped
- Lower surface is TR breaking gapped
- Edge has gapless mode described by CFT

Summary of topological order

- N=2r (even)

$\mathbf{x}$	r even				r odd		
$\mathbf{x}$	1	ψ	s_+	s_-	1	ψ	σ
$d_{\mathbf{x}}$	1	1	1	1	1	1	$\sqrt{2}$
$h_{\mathbf{x}}$	0	$1/2$	$r/16$	$r/16$	0	$1/2$	$r/16$

- N = 9+2r (odd)

$\mathbf{x}$	1	α_+	γ_+	β	γ_-	α_-	f
$d_{\mathbf{x}}$	1	$\sqrt{2+\sqrt{2}}$	$1+\sqrt{2}$	$\sqrt{4+2\sqrt{2}}$	$1+\sqrt{2}$	$\sqrt{2+\sqrt{2}}$	1
$h_{\mathbf{x}}$	0	$\frac{3+2r}{32}$	$\frac{1}{4}$	$\frac{15+2r}{32}$	$\frac{3}{4}$	$\frac{3+2r}{32}$	$\frac{1}{2}$

r even

$\mathbf{x}$	1	α_+	γ_+	β	γ_-	α_-	f
$d_{\mathbf{x}}$	1	$\sqrt{2+\sqrt{2}}$	$1+\sqrt{2}$	$\sqrt{4+2\sqrt{2}}$	$1+\sqrt{2}$	$\sqrt{2+\sqrt{2}}$	1
$h_{\mathbf{x}}$	0	$\frac{15+2r}{32}$	$\frac{1}{4}$	$\frac{3+2r}{32}$	$\frac{3}{4}$	$\frac{15+2r}{32}$	$\frac{1}{2}$

r odd

Relative Tensor Product Structure

- $$G_M \boxtimes_b G_N \cong G_{M+N}$$

$$G_N = \begin{cases} SO(r)_1, & \text{for } N = 2r \\ SO(3)_3 \boxtimes_b SO(r)_1 & \text{for } N = 9 + 2r \end{cases}$$

- E.g.

$so(3)_3$

$$\{1^I, \alpha_+^I, \gamma_+^I, \beta^I, \gamma_-^I, \alpha_-^I, f^I\}$$

$$\{0, \frac{3}{32}, \frac{1}{4}, \frac{15}{32}, \frac{3}{4}, \frac{35}{32}, \frac{3}{2}\}$$

$so(3)_3$

$$\{1^{II}, \alpha_+^{II}, \gamma_+^{II}, \beta^{II}, \gamma_-^{II}, \alpha_-^{II}, f^{II}\}$$

$$so(3)_3 \otimes so(3)_3$$

Condense bosons

$$\{f^I f^{II},\ \gamma_+^I \gamma_-^{II},\ \gamma_-^I \gamma_+^{II}\}$$

$$so(3)_3 \boxtimes_b so(3)_3 \equiv so(9)_1$$

32-fold Structure

The 16-fold anyon structure is extended to a 32-fold periodicity of the surface topological order.

$$G_{N+32} \cong G_N$$

This means combining 32 copies of topologically ordered surface gives us a topologically trivial surface.

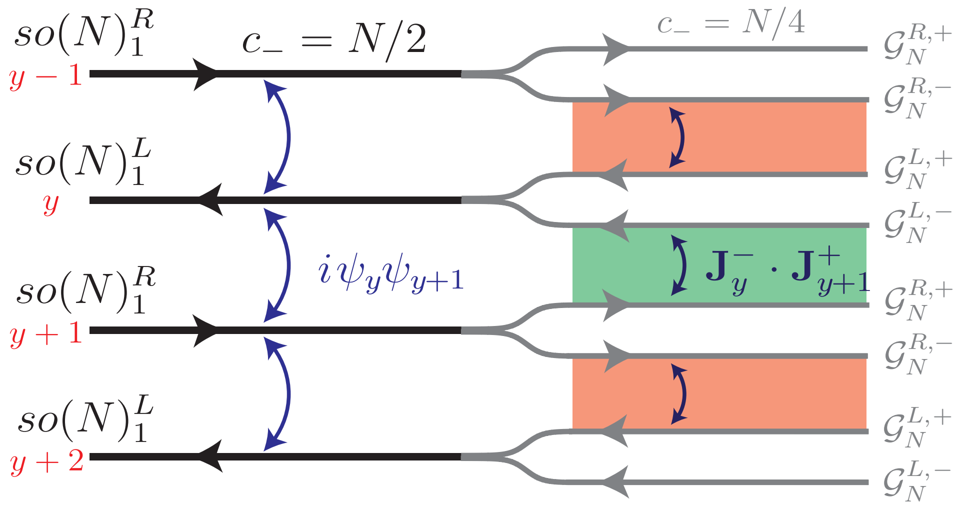
$$so(3)_3$$

$so(16)_1 \subseteq E_8$ is Topologically trivial

Conclusion

- We have a coupled wire description of the surface.

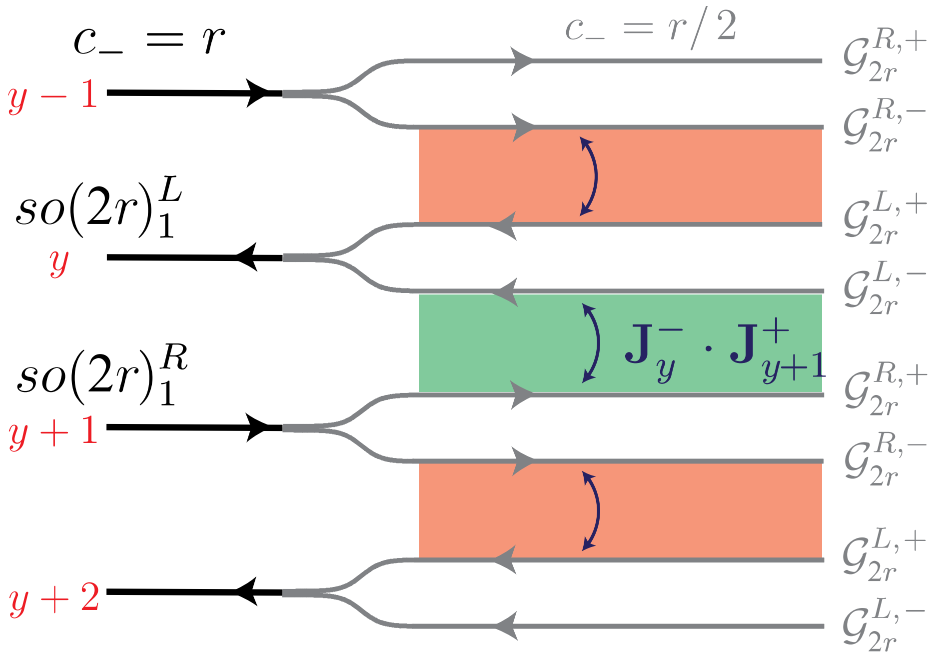
- Interaction term between fractionalized current can open a gap in the spectrum without breaking time reversal symmetry. This term is four fermion type.



- The resulting surface topological order has 32-fold periodicity.

Thank you!

Gapping even Majorana's



$$so(2r)_1 \supseteq \mathcal{G}_{2r}^+ \times \mathcal{G}_{2r}^- = so(r)_1^+ \times so(r)_1^-$$

$$\psi^1, \ldots, \psi^r \quad \psi^{r+1}, \ldots, \psi^{2r}$$

$$\mathcal{H}_{\rm int} = u \sum_{y=-\infty}^{\infty} \sum_{1 \le a < b \le r} \psi_y^{r+a} \psi_y^{r+b} \psi_{y+1}^a \psi_{y+1}^b$$

- Symmetric under anti-ferrormagnetic time reversal

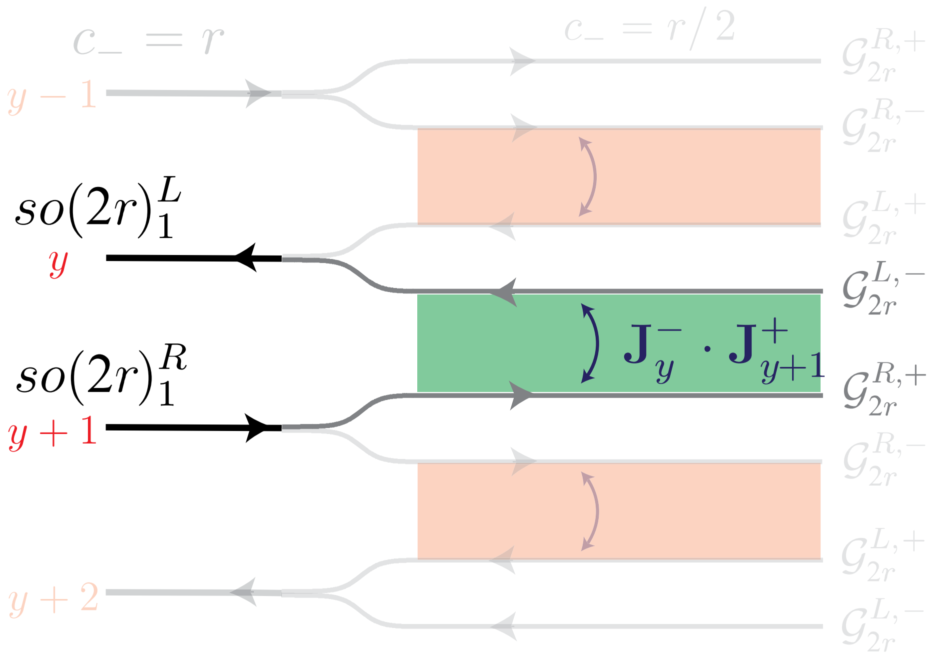
$$\mathcal{T} : \psi_y \to (-1)^y \psi_{y+1}$$

- Marginally relevant

$$\frac{du}{dl} = +4\pi(r-2)u^2$$

Gapping even Majorana's

$N = 2r = 4n$



$O(r)$ Gross-Neveu (GN) model

$$\mathcal{H}_{\mathrm{GN}} = -\frac{u}{2}\left(\boldsymbol{\psi}_R \cdot \boldsymbol{\psi}_L\right)^2$$

$$\psi_y^{r+a} = \psi_R^a \text{ and } \psi_{y+1}^a = \psi_L^a, \text{ for } a = 1, \dots, r.$$

Bosonization

$$c_{R/L}^j = (\psi_{R/L}^{2j-1} + i\psi_{R/L}^{2j})/\sqrt{2} \sim e^{i\widetilde{\phi}_{R/L}^j}$$

$$j = 1, \dots, n$$

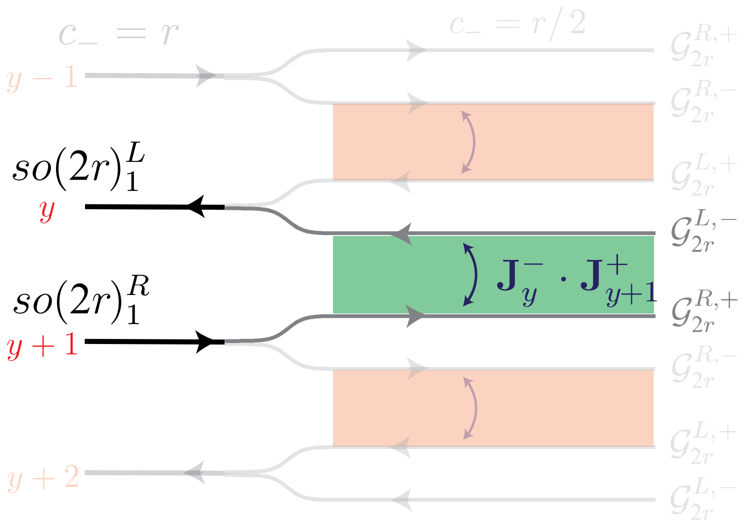
$$\mathcal{H}_{GN} \sim u\sum_{j=1}^{n} \partial_x \widetilde{\phi}_R^j \partial_x \widetilde{\phi}_L^j - u\sum_{j_1 \neq j_2}\sum_{\pm} \cos\left(2\Theta^{j_1} \pm 2\Theta^{j_2}\right)$$

$$= u\sum_{j=1}^{n} \partial_x \widetilde{\phi}_R^j \partial_x \widetilde{\phi}_L^j - u\sum_{\boldsymbol{\alpha}\in\Delta} \cos\left(\boldsymbol{\alpha}\cdot 2\boldsymbol{\Theta}\right)$$

where $2\boldsymbol{\Theta} = (2\Theta^1, \dots, 2\Theta^n)$ and $2\Theta^j = \widetilde{\phi}_R^j - \widetilde{\phi}_L^j$ $\quad \langle 2\Theta^j(x)\rangle = \pi m_\psi^j, \quad m_\psi^j \in \mathbb{Z}.$

Gapping even Majorana's

$N = 2r = 4n + 2$



$O(r)$ Gross-Neveu (GN) model

$$\mathcal{H}_{\rm GN} = -\frac{u}{2}\left(\boldsymbol{\psi}_R \cdot \boldsymbol{\psi}_L\right)^2$$

$$\psi_y^{r+a} = \psi_R^a \text{ and } \psi_{y+1}^a = \psi_L^a, \text{ for } a = 1, \ldots, r.$$

$$\mathcal{H}_{\rm GN} \sim -u \sum_{\boldsymbol{\alpha} \in \Delta_{so(2n)}} \cos\left(\boldsymbol{\alpha} \cdot 2\boldsymbol{\Theta}\right) - u\left[\sum_{j=1}^{n} \cos\left(2\Theta^j\right)\right] i\psi_R^r \psi_L^r$$

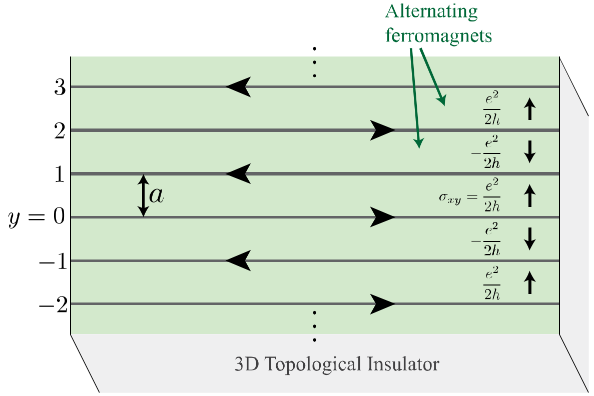
where $2\boldsymbol{\Theta} = (2\Theta^1, \ldots, 2\Theta^n)$ and $2\Theta^j = \widetilde{\phi}_R^j - \widetilde{\phi}_L^j$

$$\left\langle 2\Theta^j(x)\right\rangle = \pi m_\psi^j, \quad m_\psi^j \in \mathbb{Z}.$$

$$\mathcal{H}_{\rm GN} \sim -2n(n-1)u - nu(-1)^{m_\psi} i\psi_R^r \psi_L^r$$

Coupled wire model of TI surface

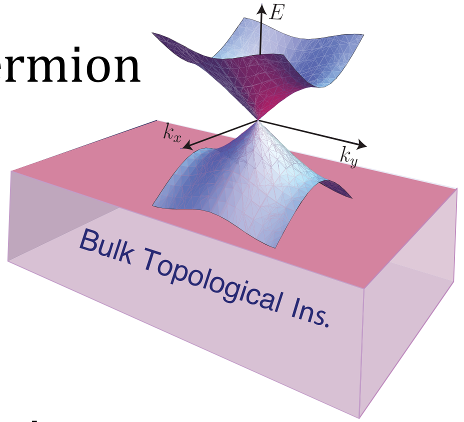
D. F. Mross, A. Essin, and J. Alicea, Phys. Rev. X **5**, 011011 (2015)

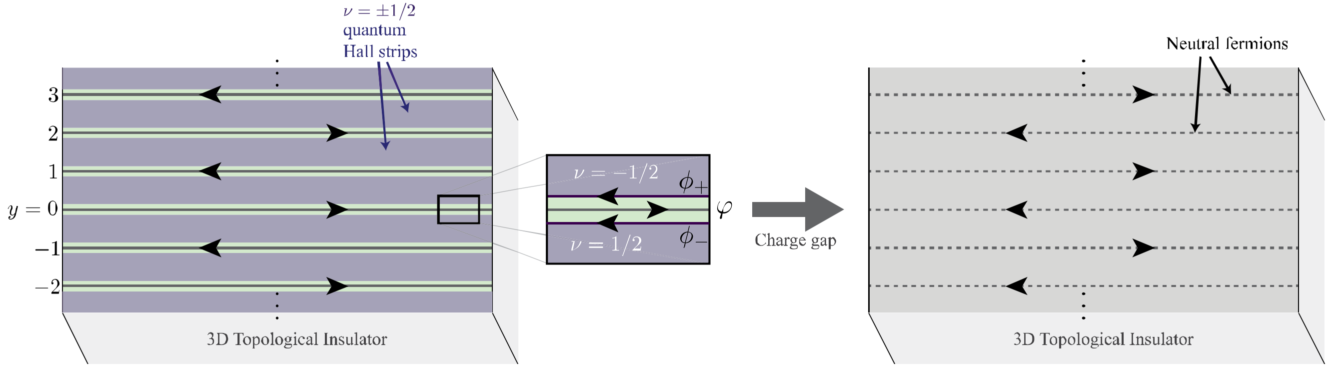


$$H = \sum_y (-1)^y \int_x \left[-i\hbar v_x \psi_y^\dagger \partial_x \psi_y - t(\psi_y^\dagger \psi_{y+1} + \text{H.c.}) \right].$$

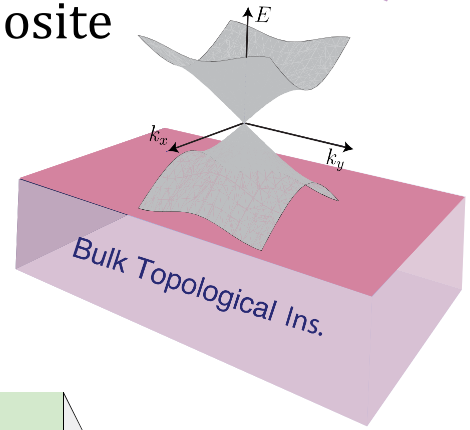
$$\tilde{\mathcal{T}} : \psi_y \to (-1)^y \psi_{y+1}.$$

charged Dirac fermion

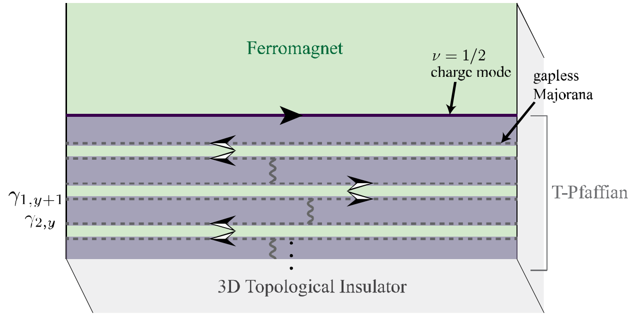




neutral composite Dirac liquid



Gapped by pairing Majorana modes .
T-pfaffian has both charge and TR symmetry



Future Plans

- T-Pffafian state -> TR symmetric topological surface state of Topological Insulator

Filling fraction, $\nu=1/2$

T-Pfaffian = Ising (spin) x $U(1)_8$ (charge)

(I,ψ,σ) $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$

- TR symmetric surface states for Fractional Topological Insulator ?

Topological order = Ising (spin) x Z_3 guage x $U(1)_8$ (charge)

Filling fraction, $\nu=1/6$

Future Plans

Kramer's Theorem -> T^2 phase -1 is doublet and +1 is singlet.

$\psi 4$ has spin ½ and phase -1 under T^2 -> electron

Anomalous 2D topological state

/4 is a fermionic Kramer's singlet as it has spin ½ and T^2 phase -1 .

$\psi 0$ is a bosonic Kramer's doublet as it has spin 0 and T^2 phase +1 .

How T^2 transforms anyons in other gapped TR symmetric surface ?