# Orbital orders and orbital order driven quantum criticality

Zohar Nussinov

Washington University in St. Louis

C. D. Batista, LANL arXiv:cond-mat/0410599 (PRB)

M. Biskup, L. Chayes, UCLA; J. van den Brink, Dresden
arXiv:cond-mat/0309691(Comm Math Phys) ;
0309692 (EPL)

E. Fradkin, UIUC arXiv:cond-mat/0410720 (PRB)

G. Ortiz, E. Cobanera, Indiana arXiv:cond-mat/0702377,
0801.4391, 0812.4309, 0907.0733
(Annals of Physics, EPL, PRB); PNAS 2009

# Conclusions (new results)

- Orbital systems can order by **thermal** "order out of disorder" **fluctuations** even in their classical limit (no (1/S) zero point quantum fluctuations are necessary).
- Similar to charge and spin driven quantum critical behavior, it is theoretically possible to have **orbital order driven quantum critical** behavior. (Prediction.)
- Orbital systems can exhibit topological order and **dimensional reductions** due to their unusual symmetries (*exact or approximate*).
- A new approach to dualities.
- **Orbital nematic orders** (from symmetry selection rules) and related selection rules
- **Orbital Larmor effects** are predicted- periodic changes in the orbital state under the application of uniaxial strain.

1. What are orbital orders?

2. Models for orbital order

*"Order by disorder" in orbital systems*

3. Orbital order driven quantum criticality and glassiness

*Exact solutions as a theoretical proof of concept*

4. Symmetries and topological order

*Low dimensional gauge like symmetries and dimensional reductions; experimentally testable selection rules*

1. What are orbital orders? (old)

2. Models for orbital order (old)

*"Order by disorder" in orbital systems (thermal fluctuations) (new)*

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*Exact solutions as a theoretical proof of concept (new)*

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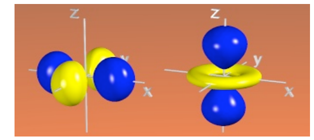
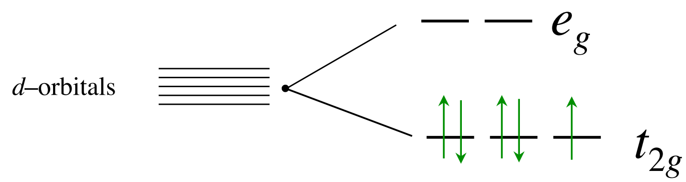
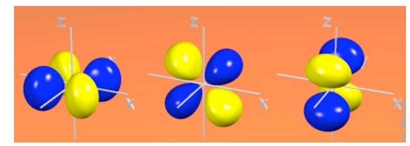
*Low dimensional gauge like symmetries and dimensional reductions; experimentally testable selection rules (new)*

# Transition Metal Compounds

- Levels in 3*d* shell split by crystal field.

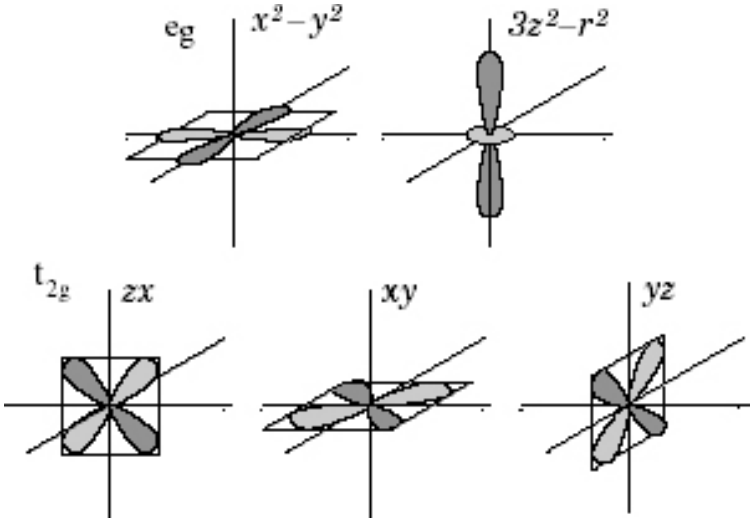


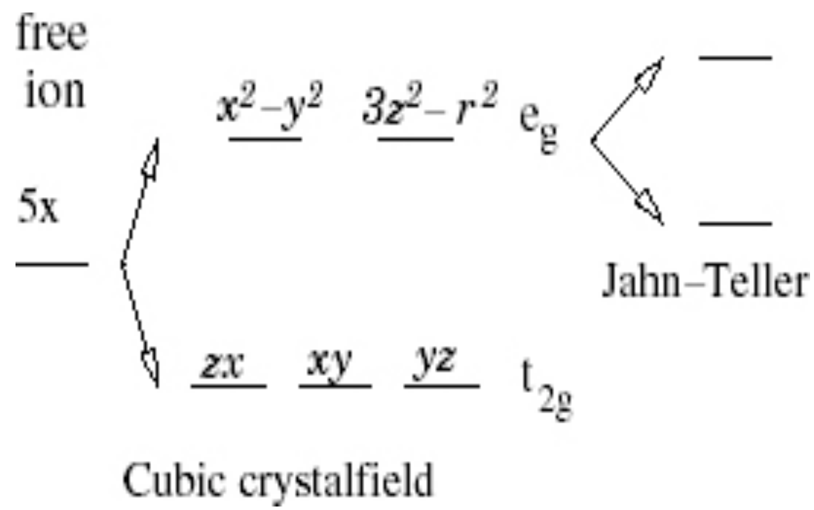


- Single itinerant electron @ each site
with multiple *orbital* degrees of freedom.

# The 3d orbitals




The five 3d orbital states share the same radial Function. Their angular dependence:

$$\left|x^2-y^2\right\rangle=\left(\frac{Y_2^2+Y_2^{-2}}{\sqrt{2}}\right)\qquad\left|3z^2-r^2\right\rangle=Y_2^0$$

$$\left|xy\right\rangle=\left(\frac{Y_2^{-2}-Y_2^2}{\sqrt{2}}\right)\qquad\left|yz\right\rangle=\left(\frac{Y_2^{-1}+Y_2^1}{\sqrt{2}}\right)\qquad\left|zx\right\rangle=\left(\frac{Y_2^{-1}-Y_2^1}{\sqrt{2}}\right)$$

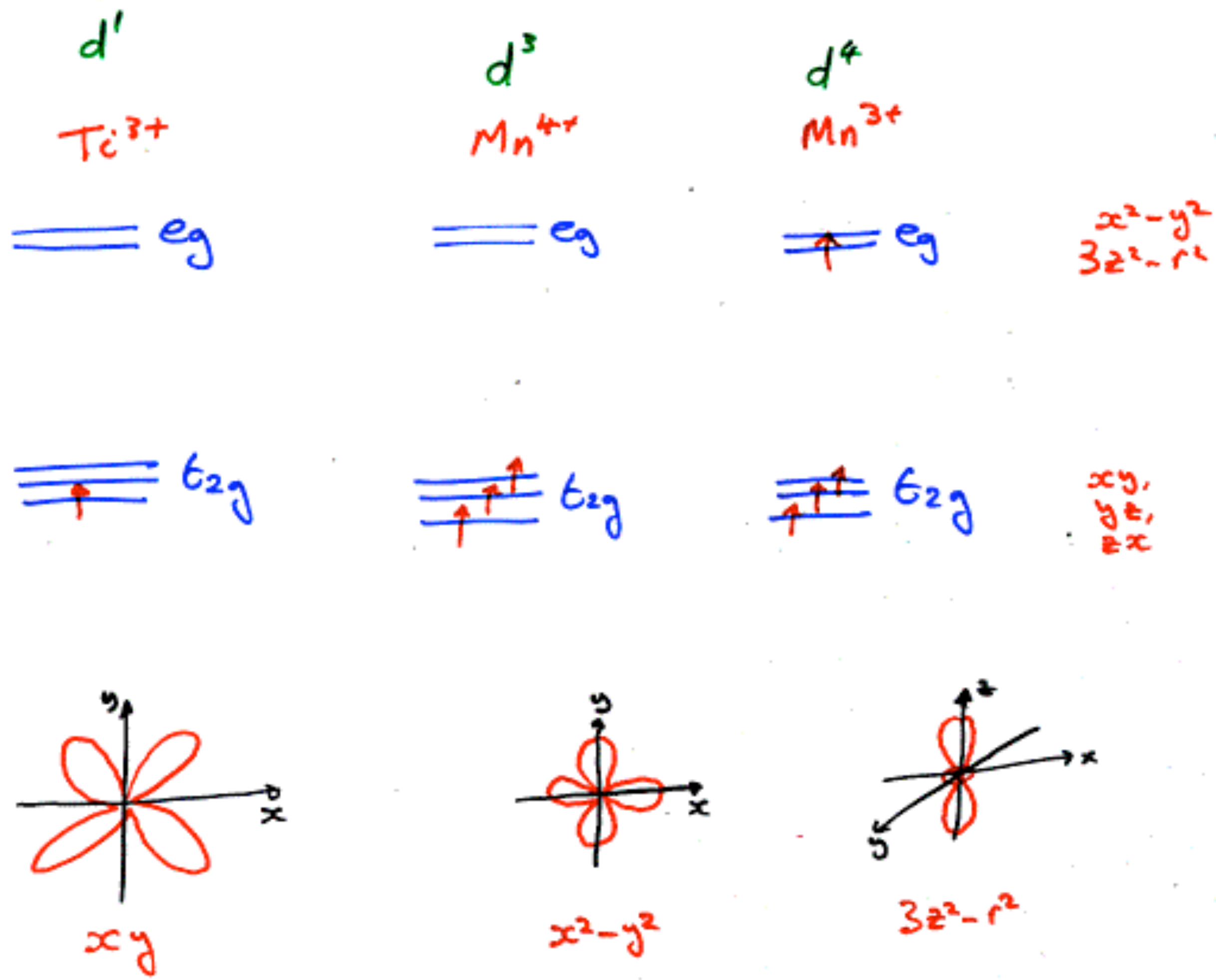
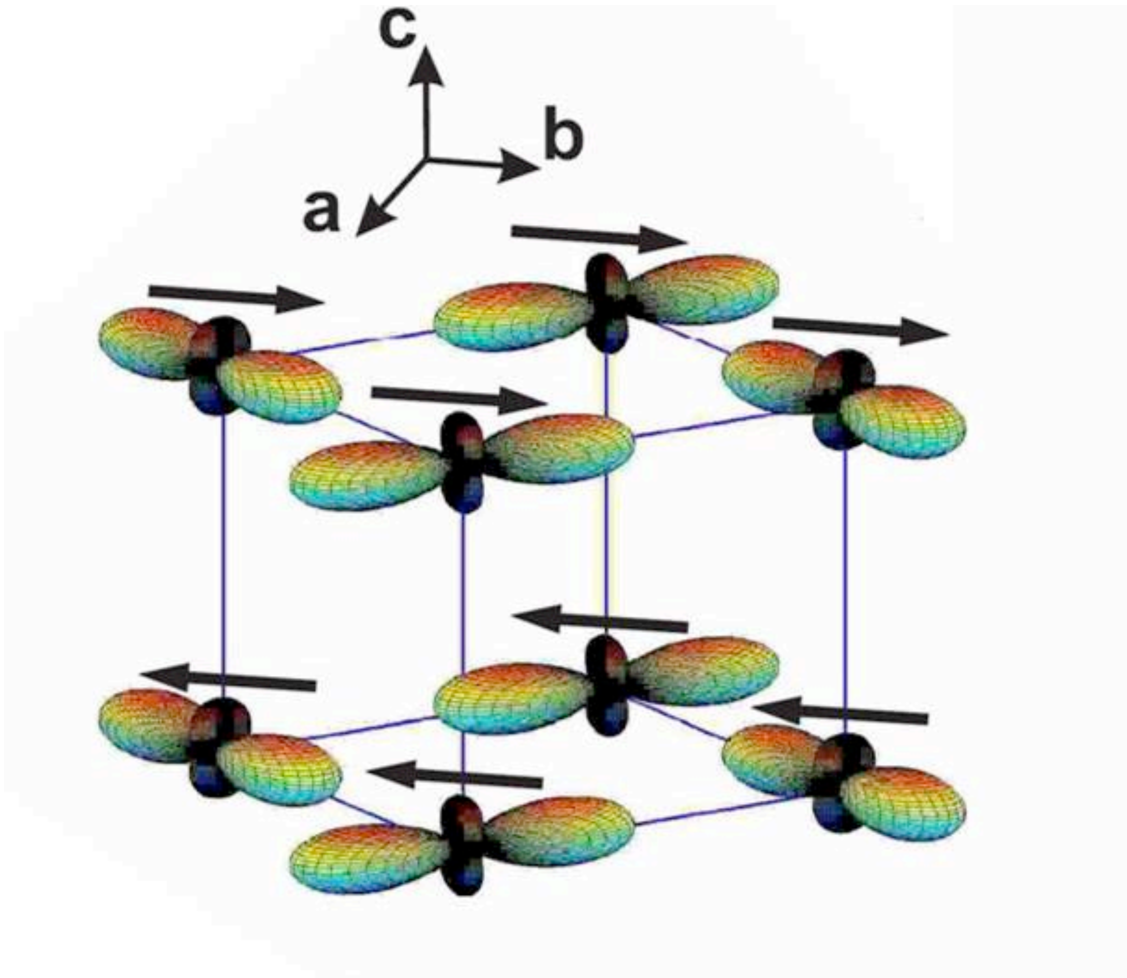
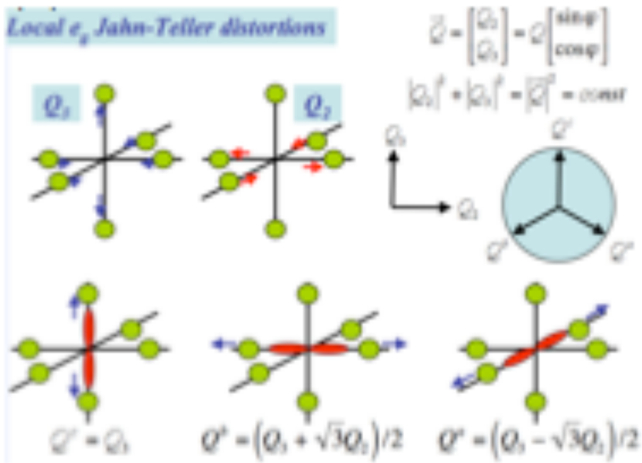



Illustration by R. Hill




$LaMnO_3$




$$\left|x^2-y^2\right\rangle = \left|S=-m=1/2\right\rangle \equiv \left|\Downarrow\right\rangle; \quad \left|3z^2-r^2\right\rangle = \left|\Uparrow\right\rangle$$

$$|xy\rangle = |S=1, m=0\rangle; |yz\rangle = 2^{-1/2}(|11\rangle + |1-1\rangle);$$
$$|zx\rangle = -i2^{-1/2}(|11\rangle - |1-1\rangle)$$

The Hilbert space of the $e_g$ orbitals is spanned by two states. The associated Jahn-Teller distortions can be expressed as vectors on a two dimensional unit disk (linear combinations of the two independent distortions $Q_{2,3}$). An effective pseudo-spin S=1/2 (or $CP_1$) representation. There is an angle of <u>*120 degrees*</u> between the three different cubic lattice symmetry related orbitals.

Similarly, the three $t_{2g}$ orbitals can be represented by an effective S=1 representation. (In a Bloch sphere representation, there is an angle of <u>*90 degrees*</u> between different point group symmetry related distortions.)

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*"Order by disorder" in orbital systems (thermal fluctuations) (new)*

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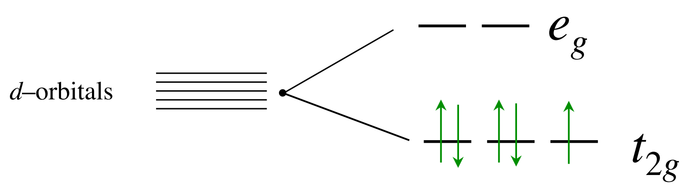
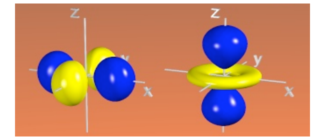
*Low dimensional gauge like symmetries and dimensional reductions; experimentally testable selection rules (new)*

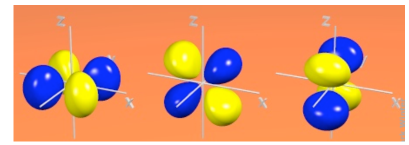
*Unlike spins, orbitals live in real space.*
*The orbital interactions*
*are <u>not</u> isotropic. Reduced symmetry*
*and frustration.*

# Transition Metal Compounds

- Levels in 3$d$ shell split by crystal field.





- Single itinerant electron @ each site with multiple *orbital* degrees of freedom.

Super–exchange approximation (and neglect of strain–field induced interactions among orbitals):

$$H = \sum_{<r,r'>} H_{\text{orb}}^{r,r'} (\mathbf{s}_r \cdot \mathbf{s}_{r'} + \tfrac{1}{4})$$

$$H_{\text{orb}}^{r,r'} = J[4\hat{\pi}_r^{\alpha}\hat{\pi}_{r'}^{\alpha} - 2\hat{\pi}_r^{\alpha} - 2\hat{\pi}_{r'}^{\alpha} + 1]$$

$\alpha$ = direction of bond $r - r'$

[Kugel–Khomskii Hamiltonian]

120°–model ($e_g$–compounds)

$V_2O_3$, $LiVO_2$, $LaVO_3$,$LaMnO_3$, …

$$\hat{\pi}_r^x = \tfrac{1}{4}(-\sigma_r^z + \sqrt{3}\sigma_r^x) \quad \hat{\pi}_r^y = \tfrac{1}{4}(\sigma_r^z - \sqrt{3}\sigma_r^x)$$

$$\hat{\pi}_r^z = \tfrac{1}{2}\sigma_r^z$$

orbital compass–model ($t_{2g}$–compounds)

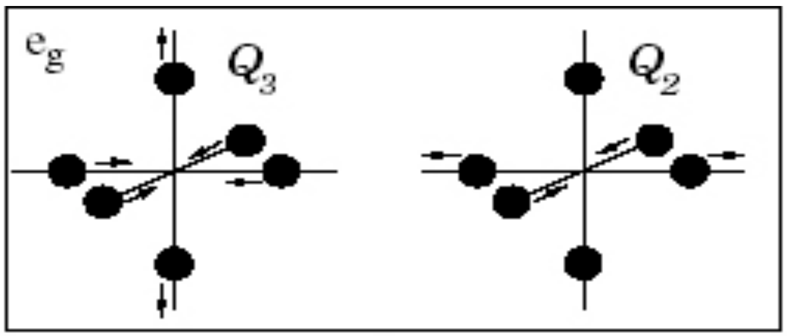
$LaTiO_3$, …

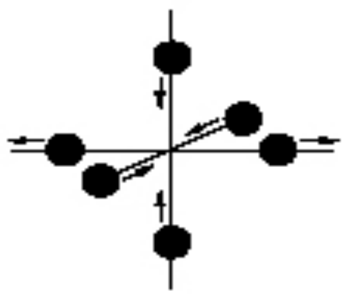
$$\hat{\pi}_r^x = \tfrac{1}{2}\sigma_r^x \qquad \hat{\pi}_r^y = \tfrac{1}{2}\sigma_r^y$$

$$\hat{\pi}_r^z = \tfrac{1}{2}\sigma_r^z$$

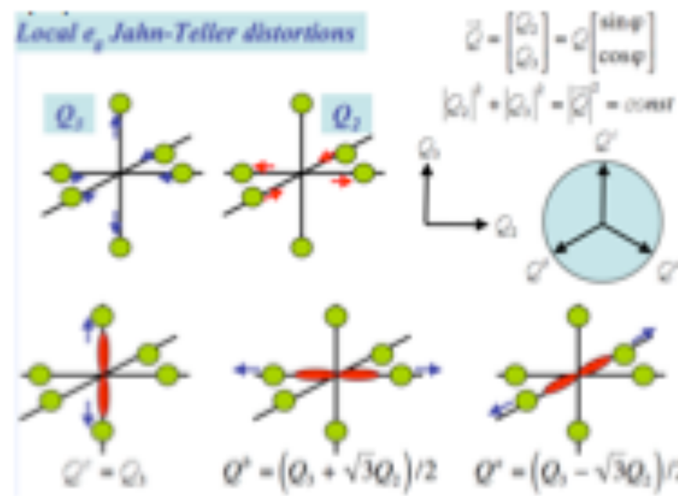
# Jahn-Teller distortions

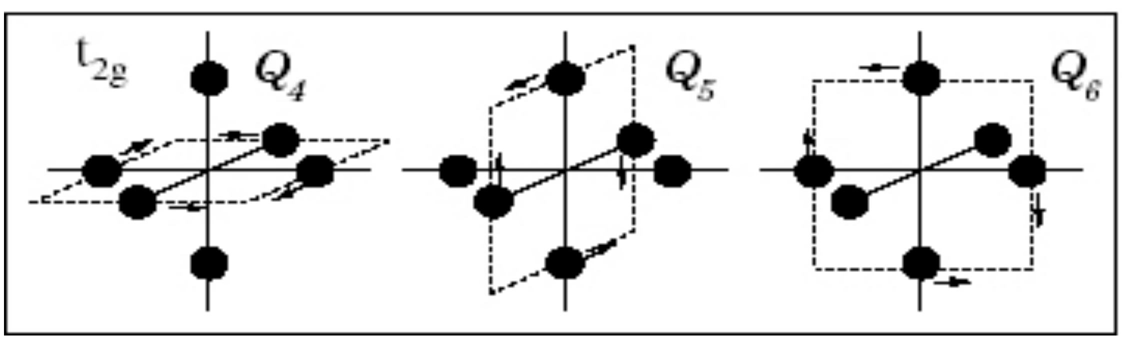
The distortions preferred by different orbital states:



$Q_3\, cos\phi + Q_2\, sin\phi$




The JT distortions can be denoted in terms of the spinor representation of the orbital states

# The orbital only interactions

The orbital component of the orbital dependent super-exchange as well as the direct Jahn-Teller orbital only interactions have a similar form:

$$H_{orb} = J \sum_{\alpha} \sum_{r} \pi_r^{\alpha} \pi_{r+e_{\alpha}}^{\alpha}$$

- Orbital only approximation: Neglect spin degrees of freedom.

120° Hamiltonian:

$$H = J\sum_r (S_r^{[a]}S_{r+e_x}^{[a]} + S_r^{[b]}S_{r+e_y}^{[b]} + S_r^{[c]}S_{r+e_z}^{[c]})$$

$\vec{S}_r$ an $XY$–spin

$S_r^{[a]} = \vec{S}_r \cdot \hat{a}$

similarly for $S_r^{[b]}$ & $S_r^{[c]}$,

$\hat{a}$, $\hat{b}$ and $\hat{c}$

unit vectors spaced @ 120°.

Orbital compass Hamiltonian:

$$H = J\sum_r (S_r^{[x]}S_{r+e_x}^{[x]} + S_r^{[y]}S_{r+e_y}^{[y]} + S_r^{[y]}S_{r+e_z}^{[z]})$$

$\vec{S}_r = (S_r^{[x]}, S_r^{[y]}, S_r^{[z]})$

– usual Heisenberg spins.

# The 120 degree model

$\vec{S}_r \in S_1$, write $\vec{S}_r = \left(S_r^{[x]}, S_r^{[y]}\right)$. $\qquad S_r^{[a]} = \vec{S}_r \cdot \hat{a}$.

$$H = J \sum_{r \in \Lambda_L} (S_r^{[a]} S_{r+e_x}^{[a]} + S_r^{[b]} S_{r+e_y}^{[b]} + S_r^{[c]} S_{r+e_z}^{[c]})$$

$$= -\frac{J}{2} \sum_{r \in \Lambda_L} \left( (S_r^{[a]} - S_{r+e_x}^{[a]})^2 + (S_r^{[b]} - S_{r+e_y}^{[b]})^2 + (S_r^{[c]} - S_{r+e_z}^{[c]})^2 \right) + \text{ constant.}$$

Attractive couplings (ferromagnetic).

Couples in *x*–direction with projection along *a*–component.
Couples in *y*–direction with *b*–component.
Couples in z- direction with c-component

Clear: Any constant spin–field is a classical ground state. Ditto for the orbital compass model.

•*U(1)* symmetry emerges in the ground state sector of the large S theory*

# Naïve spin-wave theory is a complete disaster

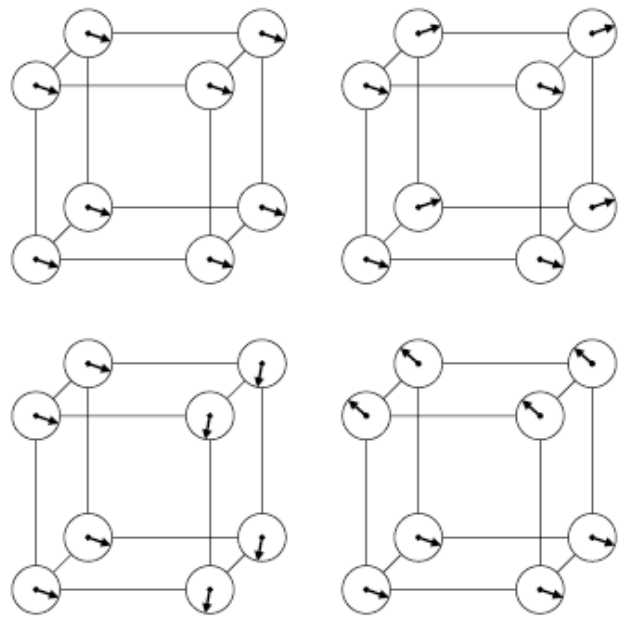
$$G(k,\omega=0) \quad \alpha \quad \frac{\Delta_a+\Delta_b+\Delta_c}{\Delta_a\Delta_b+\Delta_a\Delta_c+\Delta_b\Delta_c}$$

Fix $k_z$ $\quad G(k,\omega=0) \quad \alpha \quad \dfrac{1}{\Delta_a+\Delta_b}$

Very IR divergent.

# Lower Dimensional Symmetries

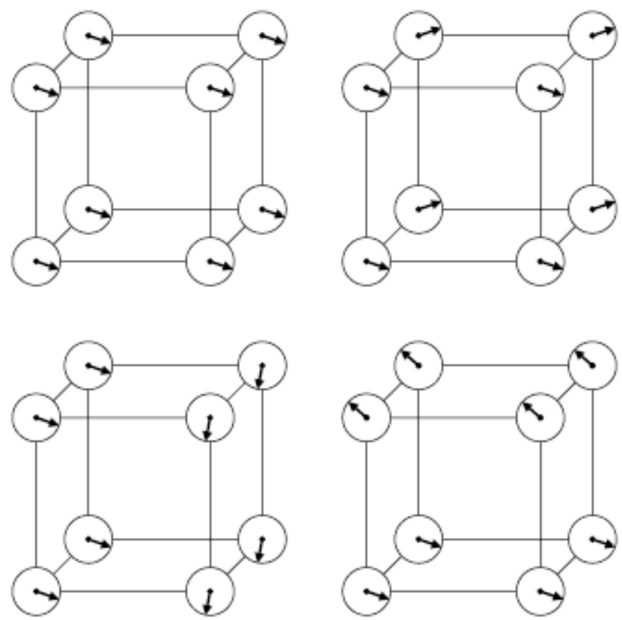
Z. Nussinov, M. Biskup,
L. Chayes, and J. v. d. Brink
0309692 (EPL)



Ising-type discrete emergent symmetries
of the classical 120 degree model

# Lower Dimensional Symmetries

L x L x L
lattice



Reflect all orbital pseudo-spins in entire planes.

Additional discrete degeneracy factor of

$$2^{3L}$$

for the ground states

# Order out of disorder-

## In the physics literature since the early 80's

J. Villain, R. Bidaux, J. P. Carton and R. Conte, *Order as an Effect of Disorder*, J. Phys. (Paris) **41** (1980), no.11, 1263–1272.

E. F. Shender, *Antiferromagnetic Garnets with Fluctuationally Interacting Sublattices*, Sov. Phys. JETP **56** (1982) 178–184 .

C. L. Henley, *Ordering Due to Disorder in a Frustrated Vector Antiferromagnet*, Phys. Rev. Lett. **62** (1989) 2056–2059.

Really clarified matters; put things on a firm foundation in a general context.

Plus infinitely many papers (mostly quantum) in which specific calculations done. Earlier orbital order work focused on zero point 1/S fluctuations.

***Our result:*** *orbital order is robust and persists for infinite S. Zero point quantum fluctuations are not needed to account for the observed orbital order.*

1) Weighting of various ground states must take into account more than just energetics:
   - Fluctuations of spins will contribute to overall statistical weight.

2) These (spin–fluctuation) degrees of freedom will themselves organize into spin–wave like modes.
   - Can be calculated (or estimated).

# Spin wave free energy calculation

Expand about the uniform state: $\theta_r = \theta^*$

$$\vartheta_r \equiv \theta_r - \theta^* \qquad H_{SW} = \frac{J}{2}\sum_{r,\alpha} q_\gamma(\theta^*)(\vartheta_r - \vartheta_{r+e_\alpha})^2$$

$$q_c(\theta^*) = \sin^2\theta^*, \quad q_{a,b}(\theta^*) = \sin^2(\theta^* \pm \frac{2\pi}{3})$$

$$\log Z(\theta^*) = -\frac{1}{2}\sum_{k\neq 0}\log\left(\sum_\alpha \beta J q_\alpha(\theta^*)\Delta(k_\alpha)\right)$$

$$\Delta(k_\alpha) = 2 - 2\cos k_\alpha$$

The free energy has strict minima at $\theta^* = n\pi/3$

Six uniform ground states: $S_r = \pm S e_\alpha$

# Stratified states: $\theta_r = (-1)^x \theta^*$

$$F(\theta^*) = \int_{k \in B.Z.} \frac{d^3k}{(2\pi)^3} \log \det(\beta J \Pi_k)$$

$$\Pi_k = \begin{pmatrix} q_1\Delta_1 + q_+\Delta_+ & q_-\Delta_- \\ q_-\Delta_- & q_1\Delta_1^* + q_+\Delta_+^* \end{pmatrix}$$

$$q_\alpha \equiv q_\alpha(\theta^*) \quad \Delta_\alpha \equiv \Delta_\alpha(k)$$

$$\Delta_\alpha^* = \Delta_\alpha(k + \pi e_\alpha)$$

$$q_\pm = \frac{1}{2}(q_2 \pm q_3)$$

$$\Delta_\pm = \Delta_2 \pm \Delta_3$$

$$F(\theta^*) > F(0), \quad \theta^* \neq 0, \pi$$

Low free energy states are not stratified.

# Finite temperature order

Reflection Positivity (chessboard estimates): $P_\beta(A) \leq \left( \frac{z_\beta(A)}{z_\beta} \right)^{B^3}$

Using Reflection Positivity along with a Peierls argument, we readily established that at sufficiently low temperatures, one of the six low free energy states is spontaneously chosen.

Interesting feature: Limiting behavior of model as $T$ goes to zero is *not* the same as the behavior of the model @ $T = 0$.

Z. Nussinov, M. Biskup, L. Chayes, and J. v. d. Brink, arXiv: cond-mat/0309692

# Nematic orbital order

For the $t_{2g}$ orbital compass type models, uniform order cannot appear. By symmetry considerations, it is established that $\langle S_r \rangle = 0$ . Instead, an "orbital nematic order" (e.g., $\left\langle (S_r^x S_{r+e_x}^x - S_r^y S_{r+e_y}^y) \right\rangle \neq 0$ in the 2D orbital compass) can be proven to onset at sufficiently low yet finite temperatures.

Z. Nussinov, M. Biskup, L. Chayes, and J. v. d. Brink, arXiv: cond-mat/0309692

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# Orbital order driven quantum criticality

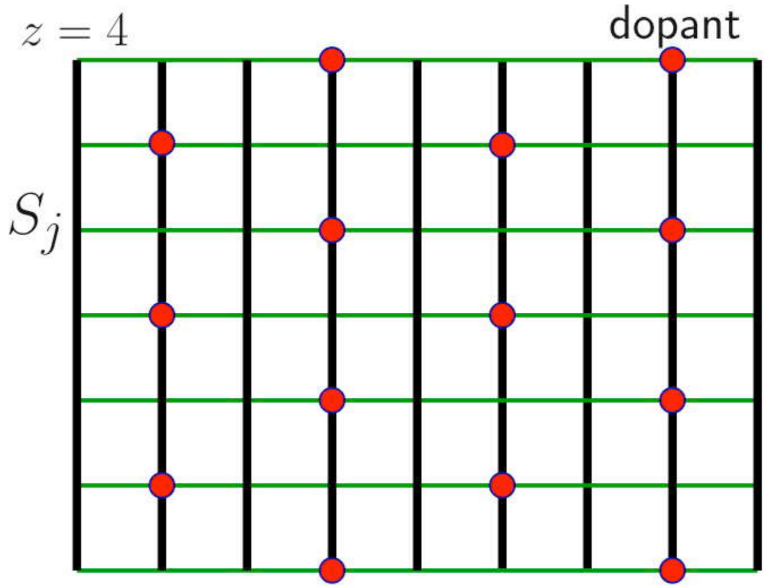
Fact: Quantum criticality can be associated with charge and spin driven orders. The transition metal oxides exhibit a rich interplay of charge/superconducting, spin, and orbital orders.

Question: Can there be an entirely new family of “orbital order driven quantum critical points”?

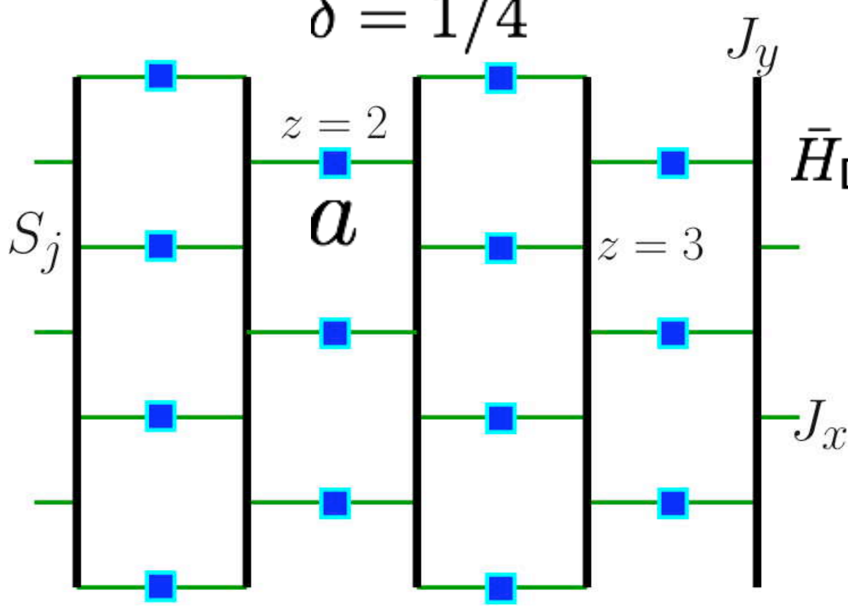
Answer: This is not forbidden and may occur theoretically. Indeed, in some simple yet exactly solvable models, there are orbital order driven quantum critical points (driven in the Hamiltonians by doping/dilution and/or uni-axial pressure).
Orbital analogues of quantum spin glasses are similarly found. For these models, the associated CFTs are standard.

Z. Nussinov and G. Ortiz, arxiv:0801.4391

# Diluted Orbital Compass Model and Criticality




$\delta = 1/4$




$$H_{\mathsf{OCM}} = -\sum_j J_\mu \sigma_j^\mu \sigma_{j+\hat{e}_\mu}^\mu$$

**After doping: New gauge symmetry**

$$\hat{O}_a = \sigma_a^x, \quad [H_{\mathsf{DOCM}}, \hat{O}_a] = 0$$

$$\bar{H}_{\mathsf{DOCM}} \equiv \hat{P}_\ell H_{\mathsf{DOCM}} \hat{P}_\ell$$

$$\hat{P}_\ell = \prod_{a=1}^{N/3} \left( \frac{\mathbb{1} + \eta_a \sigma_a^x}{2} \right) \qquad \eta_a = \pm 1$$

$$\bar{H}_{\mathsf{DOCM}} = -\sum_b \left( J_x \eta_a \, \sigma_b^x + J_y \, \sigma_b^y \sigma_{b+\hat{e}_y}^y \right)$$

$$\mathcal{Z} = \mathrm{tr}_{\mathcal{H}} e^{-\beta H_{\mathsf{DOCM}}} = 2^{N/3} \mathcal{Z}_{\mathsf{TFIM}}$$

**Quantum critical**

$(Ca_3Ru_2O_7)$

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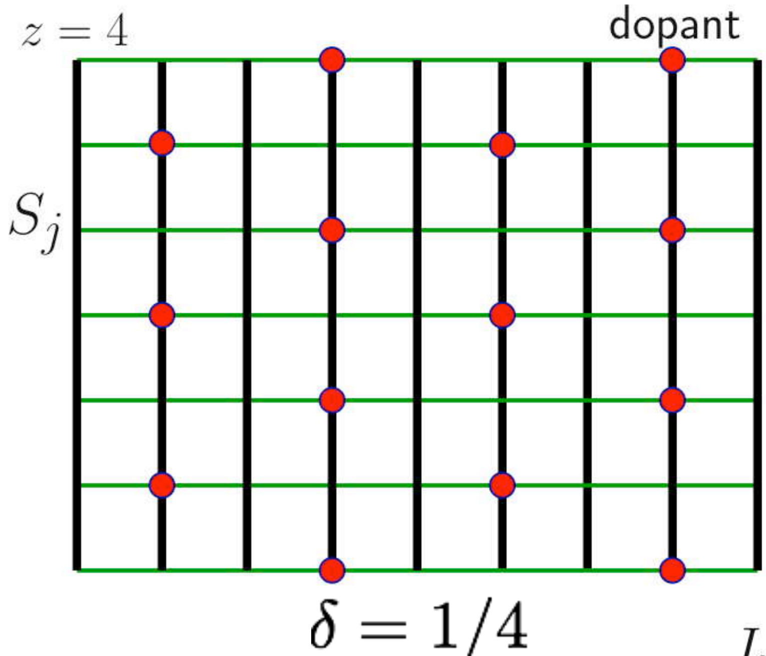
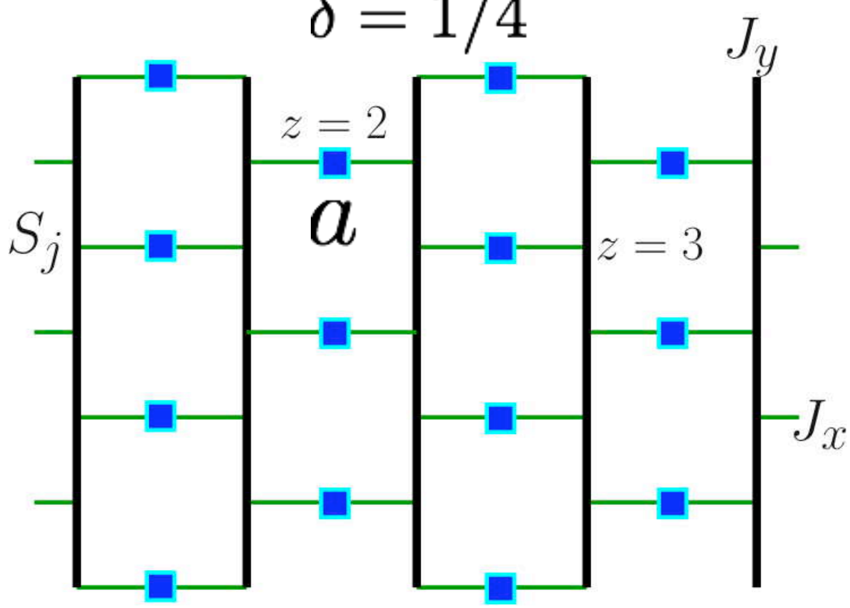


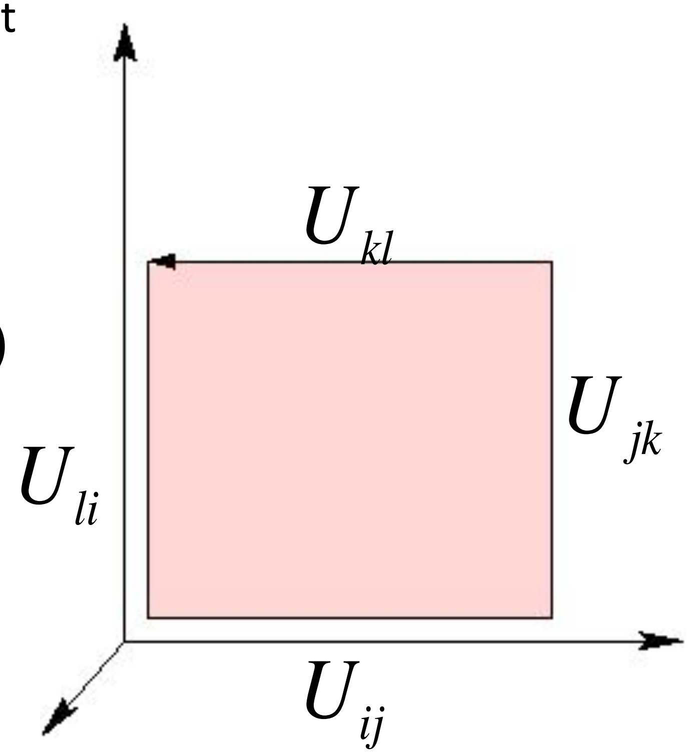

$$H_{\text{OCM}} = -\sum_j J_\mu \sigma_j^\mu \sigma_{j+\hat{e}_\mu}^\mu$$

**After doping: New gauge symmetry**

For a system with random exchange couplings $J_\mu$, replicating the same steps mutatis mutandis leads to the Random Transverse Field Ising Model. Pressure plays the role of a transverse field.

Intermezzo: using the same idea, we can solve many other models using "Bond Algebras" (Z. Nussinov and G. Ortiz, 0812.4309) and derive a new exact self duality (E. Cobanera, G. Ortiz, and Z. Nussinov 0907.0733) for $Z_N$ gauge theories in 3+1 dimensions (earlier conjectured not be self-dual). With 't Hooft ideas in mind, numerous authors studied of Wilson's action for Lattice Gauge Field Theories

$$S = -\frac{1}{g^2}\left(\sum \mathrm{Re}(Tr(U_{ij}U_{jk}U_{kl}U_{li} - 1))\right)$$




restricting the fields to $N$th roots of unity ($Z_N$).

Specifically, the dual coupling is given by

$$K_N\left(\frac{1}{2g^2}\right) \equiv K \qquad\qquad 4g_c^2 K_N\left(\frac{1}{2g_c^2}\right) = 1$$

$$\frac{1}{2}\frac{\partial F_N(K)}{\partial K} = \exp\left[-\frac{1}{2g^2}\left(1-\cos\frac{2\pi}{N}\right)\right]$$

$$F_N(K) \equiv \sum_{n=0}^{N-1} e^{2K\cos\left(\frac{2\pi n}{N}\right)} \quad .$$

Exact lattice relation! No Villain type nor any other approximation.

E. Cobanera, G. Ortiz, and Z. Nussinov 0907.0733

## The "Orbital Larmor Effect"

# Pressure effects:

$$H_P = \gamma \sum_j P_\nu \sigma_j^\nu$$

$$\frac{d\vec{\sigma}_i}{dt} = \gamma \vec{\sigma}_i \times \vec{P}_i$$

$$\vec{P}_i = P_{i,\nu} e_\nu$$

Prediction: In the presence of uniaxial pressure, the orbital state will change periodically in time.

Z. Nussinov and G. Ortiz, arxiv: 0801.4391

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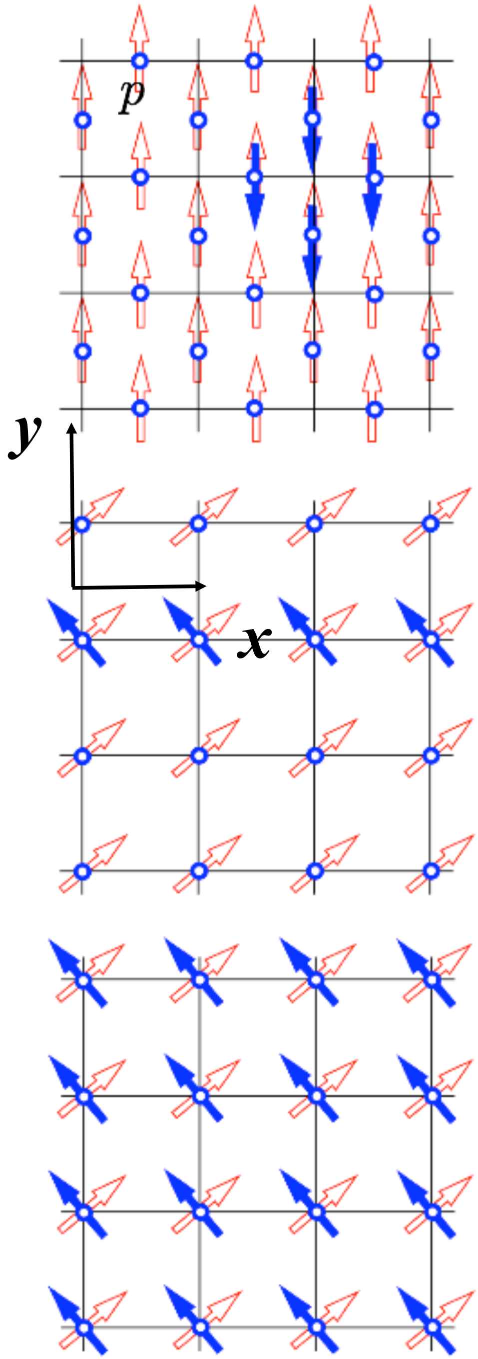
# Gauge-Like-Symmetries $D=2$




$d=0$ (Ising Gauge Theory)

$$H = -K\sum_p \sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li} \qquad G_i = \prod_{s\in nn}\sigma^x_{is}$$

$d=1$ (Orbital Compass Model)

$$H = -\sum_i [J_x\sigma^x_i\sigma^x_{i+\hat{e}_x} + J_z\sigma^z_i\sigma^z_{i+\hat{e}_y}]$$

$$O^x = \prod_{j\in C_x} i\sigma^x_j \qquad O^z = \prod_{j\in C_y} i\sigma^z_j$$

$d=D=2$ (XY model)

$$H = -J\sum_{\langle ij\rangle}[\sigma^x_i\sigma^x_j + \sigma^y_i\sigma^y_j]$$

$$U(\theta) = \prod_j \exp[-(i/2)\theta\sigma^z_j]$$

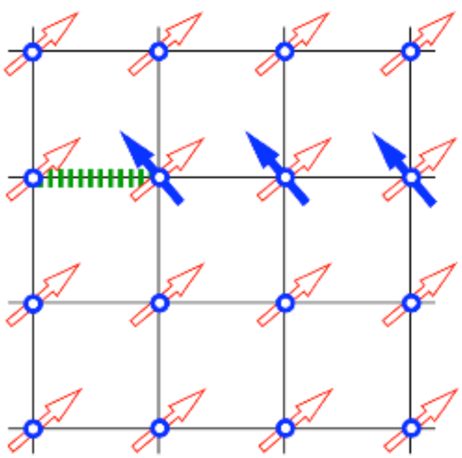
# *d*-GLSs and Topological Phases

There is a connection between Topological Phases and the group generators of *d*-GLSs and its Topological defects

$d = 1$ (*D*=2 Orbital Compass Model) $C_x$ : closed path

$$O^x = e^{i\frac{\pi}{2}\sum_{j\in C_x}\sigma_j^x} = \mathcal{P}e^{i\oint_{C_x}\vec{A}\cdot\vec{ds}}$$

Symmetries are linking operators: $O^\mu|g_\alpha\rangle = |g_\beta\rangle$



Topological defect: $C_+$ : open path

$$D^x = e^{i\frac{\pi}{2}\sum_{j\in C_+}\sigma_j^x} = \mathcal{P}e^{i\int_{C_+}\vec{A}\cdot\vec{ds}}$$

Defect-Antidefect pair creation

Z. Nussinov and G. Ortiz, PNAS (2009)

# Lower dimensional bounds

$D$-dim system with Hamiltonian $H_D$ and $d$-GLS group $\mathcal{G}_d$

The absolute value of the average of any quasi-local quantity $f$ which is not invariant under $d$-GLS $\mathcal{G}_d$ is bounded from above by the absolute value of the mean of the same quantity when this quasi-local quantity is computed with a $d$-dim $H_d$ that is globally invariant under $\mathcal{G}_d$ and preserves the range of the interactions in the original $D$-dim system

$\mathcal{C}_j$

$\eta_i$

$$\phi_i = \begin{cases} \eta_i \text{ if } i \in \mathcal{C}_j \\ \psi_i \text{ if } i \notin \mathcal{C}_j \end{cases}$$

$\psi_i$

$$|\langle f(\phi_i)\rangle_{H_D}| \leq |\langle f(\eta_i)\rangle_{H_d}|$$

**Dimensional reduction**

C. D. Batista, Z. Nussinov (cond-mat/0410599)

# To Break or not to Break

## Can we spontaneously break a *d*-GLS in a *D*-dim system ?

From the Generalized Elitzur's Theorem: (finite-range and strength interactions)

For non-$\mathcal{G}_d$-invariant quantities

- *d=0* SSB is forbidden
- *d=1* SSB is forbidden
- *d=2* (continuous) SSB is forbidden

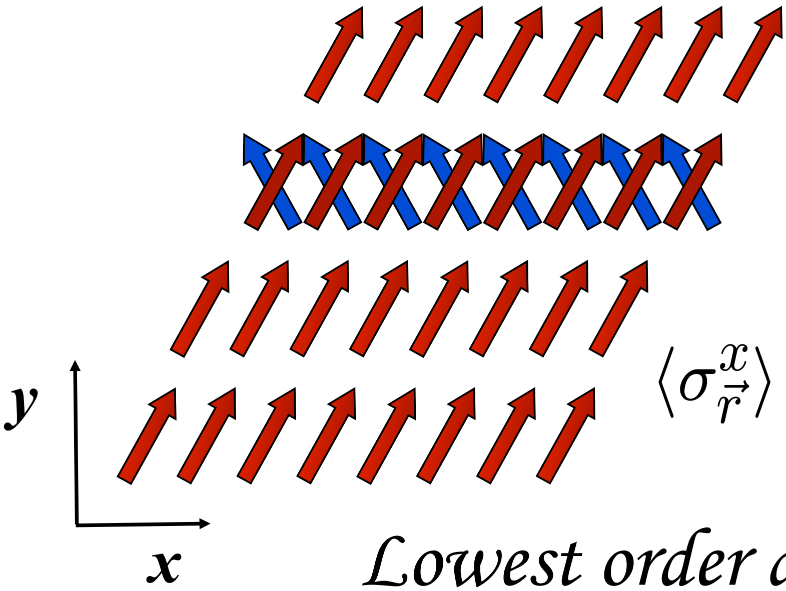
  *d=2* (discrete) SSB may be broken
- *d=2* (continuous with a gap) SSB is forbidden even at *T=0*

**Transitions and crossovers are signaled by symmetry-invariant string/brane or Wilson-like loops**

# Example of application

*Orbital Compass Model*

$$H = J \sum_{\vec{r}} (\sigma^x_{\vec{r}} \sigma^x_{\vec{r}+\hat{e}_x} + \sigma^y_{\vec{r}} \sigma^y_{\vec{r}+\hat{e}_y})$$




*Rotation by π around the y-axis*

$$\langle \sigma^x_{\vec{r}} \rangle = \langle \sigma^y_{\vec{r}} \rangle = \langle \sigma^z_{\vec{r}} \rangle = 0 \ for \ T > 0$$

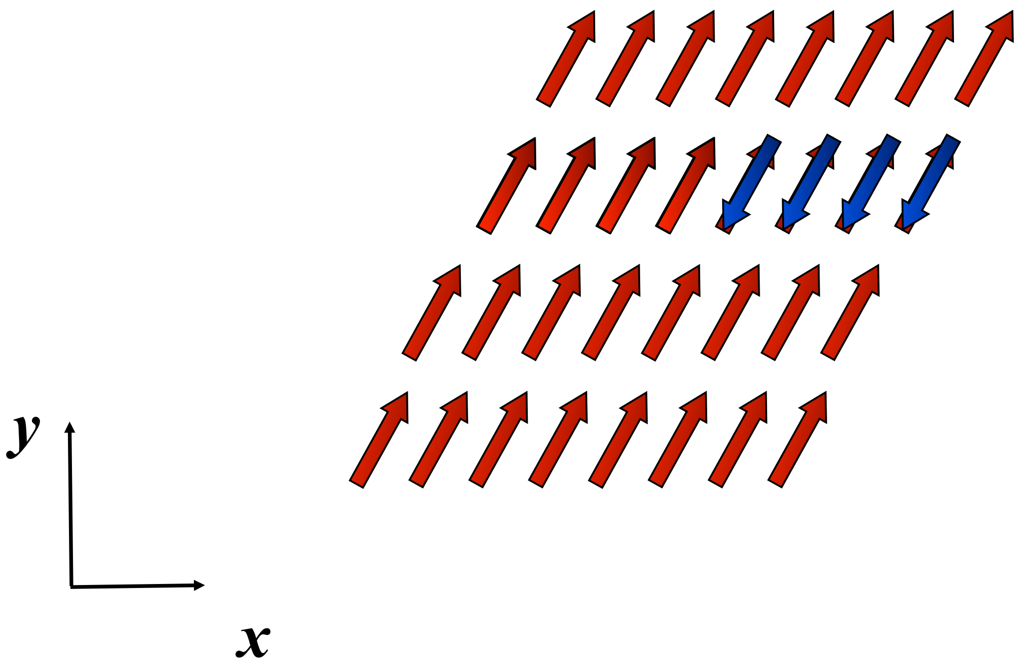
*Lowest order allowed order parameter:*

*Nematic:* $$\langle \sigma^x_{\vec{r}} \sigma^x_{\vec{r}+\hat{e}_x} - \sigma^y_{\vec{r}} \sigma^y_{\vec{r}+\hat{e}_y} \rangle$$

# Intuitive Physical Picture

*Orbital Compass Model*

$$H = J\sum_{\vec{r}}(\sigma^x_{\vec{r}}\sigma^x_{\vec{r}+\hat{e}_x} + \sigma^y_{\vec{r}}\sigma^y_{\vec{r}+\hat{e}_y})$$




*A soliton has a local energy cost.*

*2D Orbital Compass Model dual to p+ip superconducting array.* Z. Nussinov and E. Fradkin, cond-mat/0410720

# Stability and Protection of symmetries

What happens when the $d$-GLSs $\mathcal{G}_d$
are not exact symmetries of the full $H$ ?

(i.e., effect of perturbations)

Emergent Symmetries

Case I: (Exact result) Continuous $d < 2$ emergent symmetry
in a gapped system, results unchanged

Case II: Numerous systems with exact discrete $d$-GLSs
are adiabatically connected to states where
$d$-GLSs are emergent; results unchanged

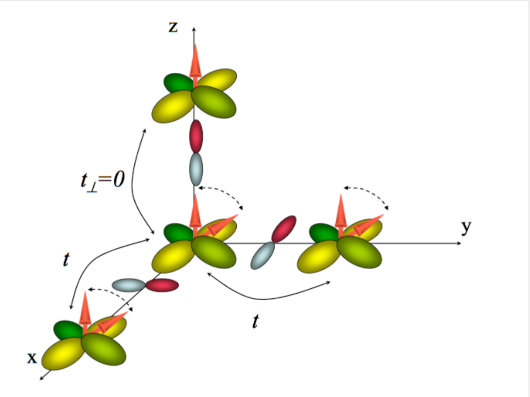
# Holographic Entropy

For independent d-GLSs
with d=1, degeneracy
is exponential in the surface area of the system.

# Symmetry based selection rules

- Kugel-Khomskii Hamiltonian $H_{KK}$ for $t_{2g}$ systems

A continuous symmetry
(A. B. Harris et al., PRL 91, 087206 (2003))



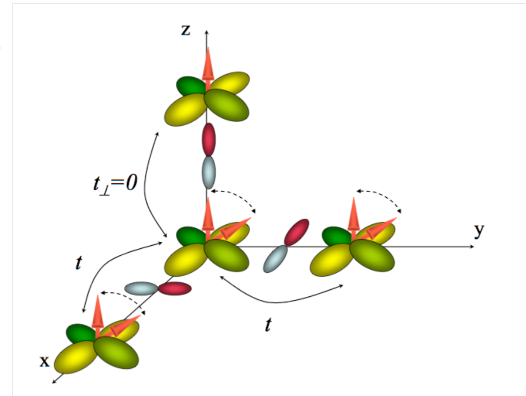

$$O_P^\gamma \equiv [\exp(i\vec{S}_P^\gamma \cdot \vec{\theta}_P^\gamma)/\hbar]$$

$$[H_{KK}, O_P^\gamma] = 0, \quad \vec{S}_P^\gamma = \sum_{r \in P} \vec{S}_r^\gamma$$

But a continuous d=2 symmetry cannot be broken, no long range order.

# Symmetry based selection rules

- Kugel-Khomskii Hamiltonian $H_{KK}$ for $t_{2g}$ systems.




For a system in |xy> state,

$$I(k_x,k_y,z,\omega)=\int dk_z e^{ik_z z} S(\vec{k},\omega)$$

vanishes for non-zero z. This is so as if two spins do not lie in the same plane (and thus have a separation along the direction orthonormal to the planes of z=0), the two point correlator is not invariant under a continuous d=2 symmetry. Other int. must be present to account for spin order. Similar considerations apply for |xz> and |yz> order. In general, if the KK interactions are dominant

$$[I(k_a,k_b,c,\omega)+I(k_b,k_c,a,\omega)+I(k_c,k_a,b,\omega)]$$

with a,b,c orthogonal axis is the largest when a,b, and c are along the crystalline axis. Nematic type parameters:

$$[2I(k_a,k_b,c,\omega)-I(k_b,k_c,a,\omega)-I(k_c,k_a,b,\omega)]$$

# Conclusions (new results)

- Orbital systems can order by **thermal** "order out of disorder" **fluctuations** even in their classical limit (no (1/S) zero point quantum fluctuations are necessary).
- Similar to charge and spin driven quantum critical behavior, it is theoretically possible to have **orbital order driven quantum critical** behavior. (Prediction.)
- Orbital systems can exhibit topological order and **dimensional reductions** due to their unusual symmetries (*exact or approximate*).
- A new approach to dualities.
- **Orbital nematic orders** (from symmetry selection rules) and related selection rules
- **Orbital Larmor effects** are predicted- periodic changes in the orbital state under the application of uniaxial strain.